Stability, Chaos Tests with Adaptive and Feedback Control Methods for 3D Discrete - Time Dynamical System

Maysoon M. Aziz* and Omar M. Jihad
Department of Mathematics, College of Computer Science and Mathematics, University of Mosul, Iraq.
*Corresponding author email id: aziz_maysoon@yahoo.com

Abstract – In this paper, a three dimensional non linear discrete-time dynamical system was introduced, the numerical solution was carried by Newton’s Raphson method. The basic properties was investigated by mean of it's fixed points, bifurcation and numerical diagrams. Stability analysis measured by eigenvalues of the characteristic equation roots ‘Lyapunov function and Jury's test which all show that the system is unstable. Chaos diagnose was calculated by Lyapunov exponent and binary test, the maximum value of Lyapunov exponent is obtain as ($L_{max} = 3.8$), Lyapunov dimension is obtain as ($D_L = 2.0872$) and binary test (0-1) is obtain as ($k = 0.9942=1$), which all shows the system is highly chaotic. Finally, the system was controlled effectively by designed feedback and adaptive controllers, the results after controls were very good system trajectories are stable and regular.

Keywords – Stability, Chaotic system, Lyapunov Function, Lyapunov Exponent, Lyapunov Dimension, Adaptive Control, Feedback Control.

I. INTRODUCTION

Dynamical system were and still deemed important role in many mathematical applications, as they were used in many different scientific fields, including : medical engineering, physical, biological, chemistry, etc., and dynamic systems consist of two types: continuous dynamic systems and discrete dynamic systems. The research dealt with discrete dynamic systems, which were distinguished from others by being more appropriate and having accurate arithmetic techniques in some scientific fields. since stability and chaos are one of the important issues in dynamic systems, the stability behavior of discrete dynamic systems and chaotic behavior have been studied in this paper, for the system to be chaotic if satisfies the following conditions. First, it must be sensitive to perturbation it’s initial conditions, whose future paths cannot be predicted, and secondly, they must not be transitive topologically, and thirdly, chaotic orbits. Third, chaotic orbits must be dense in the phase space [1]. This work was arranged as follows: Description of the system , analysis of the system and the study of the stability [2-3] of fixed points of the system using (Eigenvalues’ test, Jury test [4], and the Lyapunov function test [4]), and the numerical and practical results of the system were studied by using computer programs in Maple and MATLAB to find the roots in Newton’s Raphson numerical method and to find the stability behavior and the phase space of the system variables, and bifurcation diagrams were found [5-6]. The parameters of control in the system, and chaotic behavior [7] of the system was studied by : the Lyapunov exponent [8] and the finding of the Lyapunov dimension [8] and the binary test (0-1)[9-15], finally, feedback [16-17] and adaptive [18-20] control were performed on the system and the results of the two methods were compared.

II. SYSTEM DESCRIPTION

Three-dimensional discrete – time dynamical system [21] defined as follows : -
H : \[
\begin{aligned}
\dot{x}_{t+1} &= b_1 x_t (1 - x_t) - b_2 x_t y_t \\
\dot{y}_{t+1} &= b_3 x_t y_t - b_4 y_t z_t \\
\dot{z}_{t+1} &= b_5 y_t z_t
\end{aligned}
\] (1)

\[b_1 = 3.55, b_2 = 3.7, b_3 = 3.6, b_4 = 3.5, b_5 = 3.8\] (2)

III. ANALYSIS OF THE SYSTEM AND FINDING FIXED POINTS

In this section we will find Jacobian matrix of system (1) and define as follows:

\[
J(x,y,z) = \begin{bmatrix}
b_1(1 - 2x) - b_2 y & -b_2 x & 0 \\
b_3 y & b_3 x - b_4 z & -b_4 y \\
0 & b_5 z & b_5 y
\end{bmatrix}
\] (3)

To find the fixed points of the system, suppose \(h_1(x_t, y_t, z_t) = x_t, h_2(x_t, y_t, z_t) = y_t, h_3(x_t, y_t, z_t) = z_t\), and accordingly

\[
\begin{aligned}
b_1 x_t (1 - x_t) - b_2 x_t y_t &= x_t \\
b_3 x_t y_t - b_4 y_t z_t &= y_t \\
b_5 y_t z_t &= z_t
\end{aligned}
\] (4)

After analyzing the above system equations, we obtain the following fixed points :\( m_0 = (0,0,0)\), \( m_1 = (\frac{b_2 - 1}{b_1}, b_3, b_4)\), \( m_2 = (0, \frac{1}{b_3}, b_4, 0)\), \( m_3 = (0, \frac{1}{b_5}, \frac{1}{b_4}, 0)\), \( m_4 = (\frac{b_5 (b_5 - 1) - b_2}{b_1 b_5}, 1, \frac{b_5 (b_5 - 1) - b_2}{b_1 b_5}, \frac{1}{b_5})\).

**Lemma (1) :**

Let

\[
H(\lambda) = a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 = 0
\] (5)

A characteristic equation from (3) for the system (1), so the following cases are true:

1. If the absolute value of the eigenvalues resulting from equation (5) is greater than one, then the fixed points of the system (1) are unstable and called the source, but if one of the values at least is greater than one, then the fixed points are it's called a saddle.
2. If the absolute value of the eigenvalues resulting from equation (5) is less than one, then the fixed points of the system (1) are stable in parallel and called the sink.
3. If the absolute value of the eigenvalues resulting from equation (5) is equal to one, then the fixed points of the system (1) are called non-hyperbolic, but if there are no values equal to one, then the fixed points are called hyperbolic.

IV. STABILITY ANALYSIS

In this section we analysis the stability fixed points of the system (1) using the following tests:
4.1. Eigenvalues Test

From equations (2) and (3) substituting the point, \( m_0 = (0, 0, 0) \), we get \( J = \begin{bmatrix} 3.55 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \)

Finding the determinant \( \text{Det}(\lambda I - J) = 0 \), we get

\[ \lambda^3 - 3.55\lambda^2 = 0 \] (6)

By solving equation (6) and using Lemma (1), we obtain the following eigenvalues: \( |\lambda_1,2| = 0 \), \( |\lambda_3| = 3.55 \)

Then the fixed point \( m_0 \) is unstable, similarly, we test the rest of the points \( m_1, m_2, m_3, m_4 \), which are shown in Table (1).

<table>
<thead>
<tr>
<th>Fixed Points</th>
<th>Eigenvalues</th>
<th>Status</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_0 = (0, 0, 0) )</td>
<td>(</td>
<td>\lambda_1,2</td>
</tr>
<tr>
<td>( m_1 = (0.7183, 0, 0) )</td>
<td>(</td>
<td>\lambda_1</td>
</tr>
<tr>
<td>( m_2 = (0.27, 0.4226, 0) )</td>
<td>(</td>
<td>\lambda_1</td>
</tr>
<tr>
<td>( m_3 = (0.02631, 0.2857) )</td>
<td>(</td>
<td>\lambda_1</td>
</tr>
<tr>
<td>( m_4 = (0.444, 0.2631, 0.171) )</td>
<td>(</td>
<td>\lambda_1</td>
</tr>
</tbody>
</table>

Lemma (2) :-

Let

\[ H(\lambda) = a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 \] characteristic equation from (3) for system (1), so a Table (2) for the Jury's matrix criterion is defined as follows :

<table>
<thead>
<tr>
<th>( \lambda^0 )</th>
<th>( \lambda^1 )</th>
<th>( \lambda^2 )</th>
<th>( \lambda^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_0 )</td>
<td>( a_1 )</td>
<td>( a_2 )</td>
<td>( a_3 )</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>( a_2 )</td>
<td>( a_1 )</td>
<td>( a_0 )</td>
</tr>
<tr>
<td>( b_0 )</td>
<td>( b_1 )</td>
<td>( b_2 )</td>
<td></td>
</tr>
<tr>
<td>( b_2 )</td>
<td>( b_1 )</td>
<td>( b_0 )</td>
<td></td>
</tr>
<tr>
<td>( c_0 )</td>
<td>( c_1 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( c_1 )</td>
<td>( c_0 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Such that

\[ b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \ \text{k} = 0, 1, 2, \]

\[ c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}, \ \text{k} = 0, 1 \]

Then the fixed points are said to be stable if they satisfies the following conditions: \( H(1) > 0 \), \( (-1)^n H(-1) > 0 \), \( |a_0| < |a_n|, |b_0| > |b_{n-1}|, |c_0| > |c_{n-2}| \)
Where H is characteristic equation, otherwise, the fixed points are unstable.

4.2. Jury Test

We test the stability of the point \( m_0 = (0, 0, 0) \), from equation (6) we form Table (3) of Jury’s table as follows:

<table>
<thead>
<tr>
<th>( a_0 )</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( a_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>-3.55</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>-3.55</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>3.55</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>3.55</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-3.55</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From the lemma (2), the condition \( |c_0| < |c_{n-2}| \) It is not achieved, therefore the point \( m_0 \) is unstable, and in the same way test the rest of the fixed points \( m_1, m_2, m_3, m_4 \), which shows that unstable, so the system (1) is unstable.

Lemma (3):

Let the Lyapunov quadratic function equation of the system (1) as follows:

\[
V(x, y, z) = x^2 + y^2 + z^2 > 0
\]

Where \( x, y, z > 0 \), by using \( \Delta V \) we get:

\[
\Delta V(x, y, z) = V(x_{t+1}, y_{t+1}, z_{t+1})^2 - V(x_t, y_t, z_t)^2
\]

Then the fixed points of the system (1) are said to be stable if they are \( \Delta V \leq 0 \), otherwise the points are unstable.

4.3. Lyapunov Function Test

In this section we test the stability of the fixed points for system (1) by Lyapunov function method we test the stability of point \( m_4 = (0.444, 0.2631, 0.171) \), by substituting equations (1) and (2) into equation (7) we get:

\[
\Delta V(0.444, 0.2631, 0.171) = (3.55(0.444) - 3.55(0.444)^2 - 3.7(0.444)(0.2631))^2 + (3.6(0.444)(0.2631) - 3.5(0.2631)(0.171))^2 + (3.8(0.2631)(0.171))^2 - (0.444)^2 + (0.2631)^2 + (0.171)^2 = 0.0002 > 0
\]

Then \( m_4 \) is unstable, so the system (1) is unstable.

V. THE DYNAMIC BEHAVIOR AND NUMERICAL DIAGRAMS OF SYSTEM (1)

In this section we will study the system behaviors and some numerical results, using mathematical programs in MATLAB and Maple.

5.1. Newton's Raphson Numerical Method

In this section, Newton’s Raphson numerical method was used to find the roots of the system (1) with minimum error (0.0001), by using a written program in MATLAB with \( (x, y, z) = (0.3, 0.2, 0.1) \).

5.2. Time Behavior of System (1)
In this section, a system time series (1) was generated for 100 iterations and at the parameter values \( b_1 = 3.55, b_2 = 3.7, b_3 = 3.6, b_4 = 3.5, b_5 = 3.8 \) with values \((x, y, z) = (0.3, 0.2, 0.1)\), and shown in Figure (1), which shows us the unstable behavior of the system.

![Fig. 1. Time behavior of the system with states \(x_t, y_t, z_t\).](image1)

5.3. Phase Space of the System (1)

To find the phase space for the system variables (1), the system parameters are fixed at the values \( b_2 = 3.7, b_3 = 3.6, b_4 = 3.5, b_5 = 3.8 \) and shown in Figure (2), so when the parameter \( b_1 = 3.55 \) we notice the instability as the phase space is chaotic this chaos represents the steady state \( m_4 \).

![Fig. 2. The phase space of the system (1) with the variables \(x_t, y_t, z_t\) and \(b_1 = 3.55\).](image2)

In Figure (3), the phase space was found for some of the system states, which show us the growth and dispersion when increasing the value of the parameter \( b_1 \), as we notice the increase in chaos and a closed curve around the fixed point \( m_4 \).
In this section we will study the bifurcation diagrams of the parameters $b_1, b_3$ that are most effective in the system (1), and to find the bifurcation of the parameter $b_1$, the system parameters are fixed at the values $b_2 = 3.7$, $b_3 = 3.6$, $b_4 = 3.5$, $b_5 = 3.8$ with a change in the values of the parameter $b_1$ within the period $[2, 3.6]$ with an increase of $0.001$ and the values $(x, y, z) = (0.3, 0.2, 0.1)$ and shown in Figure(4), which shows the internal balance of the parameter within the period $[2.8, 3.1]$ and this balance represents the stability of the complex fixed point $m_4$ in the indication of its stability.
To find the bifurcation of the parameter $b_3$, the system parameters are fixed at the values $b_1 = 2.92, b_2 = 3.7, b_4 = 3.5, b_5 = 3.8$ with a change in the parameter $b_3$ within the period [2.6, 4] and by an increase of (0.001), and shown in Figure(5), which shows the internal balance of the parameter within [3.57, 3.69].

**VII. LYAPUNOV EXPONENT**

In this section we will use the Lyapunov exponent test, which is one of the indicators of divergence or convergence between paths in the state space to denote chaos, if at least one of the Lyapunov exponent values is greater than zero, and by using a mathematical program in the MATLAB, the following values of the Lyapunov exponent are obtained: $L_1 = 0.507879, L_2 = -0.176484, L_3 = 3.8$
To calculate the Lyapunov dimension we use the following formula:

\[ D_L = 2 + \frac{L_1 + L_2}{|L_3|} = 2 + \frac{0.5078794(-0.176484)}{|3.8|} = 2.0872 \]

Accordingly, System (1) is a highly chaotic system, and Figure (6) shows that.

**VIII. THE BINARY TEST (0-1)**

In this section, the binary test was used to study the chaoticity of system (1), where the system is said to be stable if \((k)\) approaches to zero and if \((k)\) approaches to one the system is said to be chaotic, and by using a written program in MATLAB. Time series for (100) iterations and the system parameter values were fixed at the values \(b_1 = 3.55, b_2 = 3.7, b_3 = 3.6, b_4 = 3.5, b_5 = 3.8\), with values \((x, y, z) = (0.3, 0.2, 0.1)\) to calculate \(q_c(t)\), \(p_c(t)\) shown in Figure (7a), which show us similar behavior to Brownian motion, and the average square displacement \(M_c(t)\) with time \((t = 100)\) and shown in Figure (7b), which shows We use the linear growth of the average displacement with time, and find the mean \((k)\) of the growth asymptotically \(K_c\) with \((c)\), where \(c \in (0, \pi)\) as it's obtain \(k = 0.9942\) approaching to one which shows us the chaotic behavior of the system (1), shown in Figure (7c).

**IX. CONTROL SYSTEM (1)**

9.1. Feedback Control Strategy

In this section, feedback control of the system (1) was performed, and by adding the control units \(u_1, u_2, u_3\) we get:-
Where $u_1, u_2, u_3$ are defined as follows:

$$
\begin{align*}
    u_1 &= b_1 x_t (-2 + x_t) + b_2 x_t y_t \\
    u_2 &= -b_3 x_t y_t + b_4 y_t z_t - y_t \\
    u_3 &= -b_5 y_t z_t - z_t 
\end{align*}
$$

Substituting equation (9) into (8) we get:

$$
\begin{align*}
    x_{t+1} &= -b_2 x_t \\
    y_{t+1} &= -y_t \\
    z_{t+1} &= -z_t 
\end{align*}
$$

Where $x_t$, $y_t$, $z_t$ are the system variables and $b_3$ is the system parameter.

### 9.1.1 Stability Test

In this section, for testing the stability system (10) we find the Jacobian matrix of system (10) as follows:

$$
J_{(x_t,y_t,z_t)} = \begin{bmatrix}
    -b_3 & 0 & 0 \\
    0 & -1 & 0 \\
    0 & 0 & -1
\end{bmatrix}
$$

And after test the fixed points $m_0, m_1, m_2, m_3, m_4$ show that stable, the results are shown in Table (4). So the system (10) is stable.

<table>
<thead>
<tr>
<th>Fixed Points</th>
<th>Eigenvalue Test</th>
<th>Jury Test</th>
<th>Lyapunov Function Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_0 = (0,0,0)$</td>
<td>Stable</td>
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<td>Stable</td>
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<td>Stable</td>
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<tr>
<td>$m_2 = (0.27, 0.4226, 0)$</td>
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<td>Stable</td>
<td>Stable</td>
</tr>
<tr>
<td>$m_3 = (0, 0.263, -0.2857)$</td>
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<td>Stable</td>
<td>Stable</td>
</tr>
</tbody>
</table>

### 9.1.2 Lyapunov Exponent of System (10)

In this section we will use the Lyapunov exponent test of system (10), and by using a mathematical program in the MATLAB, the following values of the Lyapunov exponent are obtained:

$$
L_1 = -3.550000, \quad L_2 = -1.000000, \quad L_3 = -1.000000, \quad L_4 = -1.000000.
$$

Since the values are negative, then system (10) is regular and Figure (8) shows that.

![Lyapunov exponent for system (10)](image-url)

Fig. 8. Lyapunov exponent for system (10).
9.2. Adaptive Control Strategy

In this section we use the adaptive control technique and design an adaptive control law with the unknown parameter $b_1$.

\[
\begin{align*}
x_{t+1} &= b_1 x_t (1 - x_t) - b_2 x_t y_t + u_1 \\
y_{t+1} &= b_3 x_t y_t - b_4 y_t z_t + u_2 \\
z_{t+1} &= b_5 y_t z_t + u_3
\end{align*}
\]

(11)

Where $u_1, u_2, u_3$, represent the feedback control units, which are defined as follows:

\[
\begin{align*}
u_1 &= -\alpha x_t (1 - x_t) + b_2 x_t y_t - M_1 x_t \\
u_2 &= -b_3 x_t y_t + b_4 y_t z_t - M_2 y_t \\
u_3 &= -b_5 y_t z_t - M_3 z_t
\end{align*}
\]

(12)

Where $M_1, M_2, M_3$ are constants and are positive and $\alpha$ is an approximate parameter of the parameter $b_1$, and by substituting (12) in (11) we get:

\[
\begin{align*}
x_{t+1} &= (b_1 - \alpha) x_t (1 - x_t) - M_1 x_t \\
y_{t+1} &= -M_2 y_t \\
z_{t+1} &= -M_3 z_t
\end{align*}
\]

(13)

Let the expected parameter error $\alpha$ be defined as follows:

\[e_b = b_1 - \alpha\]

(14)

Substituting (14) into (13) we get:

\[
\begin{align*}
x_{t+1} &= e_b x_t (1 - x_t) - M_1 x_t \\
y_{t+1} &= -M_2 y_t \\
z_{t+1} &= -M_3 z_t
\end{align*}
\]

(15)

9.2.1. Stability Test

In this section we test the stability of the controlled system (13). We find the Jacobian system (13) matrix as follows:

\[
\begin{bmatrix}
-0.02 + 0.04 x_t - M_1 & 0 & 0 \\
0 & -M_2 & 0 \\
0 & 0 & -M_3
\end{bmatrix}
\]

Where $\alpha = 3.57, M_1 = 0.5, M_2 = 0.7, M_3 = 0.9$, and Table (5) shows the results the stability tests.

<table>
<thead>
<tr>
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<th>Jury Test</th>
<th>Lyapunov Function Test</th>
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<td>Stable</td>
<td>Stable</td>
<td>Stable</td>
</tr>
</tbody>
</table>

9.2.2. Lyapunov Exponent of System (13)

In this section we will use the Lyapunov exponent test of system (13) and by using a mathematical program in
the MATLAB, the following values of the Lyapunov exponent are obtained: \( L_1 = -0.520000, L_2 = -0.700000, L_3 = -0.900000 \)

Since the values are negative, then system (13) is regular and figure (9) shows that.

**Fig. 9.** Lyapunov exponent for system (13).

**X. CONCLUSION**

In this work, a discrete-time three dimensional system was taken, and the stability of the fixed points of the system was studied and analyzed, and it was found that the system is stable at its fixed points, except for the point \( m_4 \), which is sometimes complicated in indicating its stability, and the time behavior and the phase space of the system variables were studied which shown the system is unstable, and found the bifurcation diagrams for the bifurcation parameters of system, and Newton’s Raphson numerical method was used to find the best roots for system with minimum error. The chaos of the system was studied using: Lyapunov exponent is obtain as \( L_1 = 0.507879, L_2 = -0.176484, L_3 = 3.8 \) and Lyapunov dimension is obtain as \( D_L = 2.0872 \), the binary test (0-1) is obtain as \( k = 0.9942 \), which shown that system (1) is chaotic. Finally, the feedback control and the adaptive control of the system were conducted, and the two methods were compared. It was found that the adaptive control method of the system is better in achieving the stability of the system at all its fixed points, whatever the values that were imposed for the system parameters, while the method of feedback control depends on the value of the control parameter \( b_1 \) in achieving stability, as the system is stable if the parameter is less than one, otherwise the system is unstable, while diagnosis chaos of the system in both cases show that the system is regular.

**REFERENCES**


[10] Xin, Baogui, and Zhiheng Wu. “Projective synchronization of chaotic discrete dynamical systems via linear state error feedback control.” Entropy 17, no. 5 , 2015, 2677-2687

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AUTHOR’S PROFILE

First Author  
Dr. Maysoon M. Aziz, Department of Mathematics, College of computer science and mathematics, Mosul University Iraq. Ph.D., Mathematical Models and Time Series, Mosul University,. M.Sc. Mathematical Statistics, School of Maps, Sussex University Brighton, England. email: Aziz_maysoon@uomosul.edu.iq

Second Author  
Omar M. Jihad, Department of Mathematics, M.Sc. Student, College of computer science and mathematics, Mosul University Iraq. email: omarmoh1992lh@gmail.com