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On the Estimation of the Generalized Inverted Rayleigh Distribution with Real Data Applications

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Abstract **– This article discusses Bayesian and non-Bayesian estimator methods to estimate the unknown parameters of the generalized inverted Rayleigh distribution (GIRD). Maximum likelihood estimators of the unknown parameters are obtained. Also, Bayesian estimators under squared error (SE) and linear-exponential (LINEX) loss functions are derived. Moreover, applications to real data sets are provided.**

Keywords **– Bayesian Estimator, Conjugate Prior, Loss Function, Maximum Likelihood Estimator.**

I. INTRODUCTION

The inverse Rayleigh distribution is one of the most flexible distributions among the inverted scale family. It has been considered as a suitable model in life testing and reliability theory. Many authors were interested in the generalized form of this distribution. A generalized inverted scale family of distributions was introduced by [1]. They generalize the scale family by introducing a shape parameter α to obtain a generalized scale family of distributions. The generalized inverted Rayleigh distribution (GIRD) is a general form of the inverse Rayleigh distribution (IRD). References [2] and [3] derived some properties of the maximum likelihood estimator for the parameter of IRD. Reference [4] derived some measures of IRD. He obtained the mean, harmonic mean, geometric mean, mode, and the median. He also estimated the unknown parameter using different methods of estimation. A comparison of the negative moment estimator with maximum likelihood estimator of the IRD was studied by [5]. Reference [6] derived the maximum likelihood estimator and Bayesian inference of the unknown parameter of the IRD based on lower record values. Moreover, some properties of the beta inverse Rayleigh distribution including quantile function, moments, mean deviation, Shannon entropies and order statistics was investigated by [7]. However, the new generalization of IRD, named GIRD, was not discussed widely in the literature. A few studies were considered the GIRD. For instance, the statistical properties of the GIRD were studied by [8]. Furthermore, step stress partially accelerated lifetime failure rate was investigated by [9].

Bayesian and non-Bayesian estimations are very important in several real life problems. Many authors made statistical inference using these two methods among them, [10]-[12]. The following studies discussed these methods. Reference [13] derived maximum likelihood and Bayesian estimators for the parameters of the exponentiated gamma distribution in the case of complete and type II censoed samples. Also, different methods of estimations have been discussed by [14]. They derived maximum likelihood estimators for complete and type II censored samples from the exponentiated gamma distribution. Also, they obtained least-squares, weighted least-squares estimators and estimators based on percentiles of the parameters. In addition, method of moment estimators and L-moment estimators were discussed. Recently, [15] obtained the maximum likelihood estimators for parameters of the modified inverse Rayleigh distribution.

In this paper, we consider the GIRD which has the following probability density function (pdf) and cumulative density function (cdf), respectively, as follows

$$f\left(x;\lambda,\alpha\right)=\frac{2\alpha}{\lambda^2x^3}e^{-(\lambda x)^{-2}}\left[1-e^{-(\lambda x)^{-2}}\right]^{\alpha-1},\ x>0,\alpha,\lambda>0, \tag{1}$$

and

$$F\left(x;\lambda,\alpha\right)=1-\left[1-e^{-(\lambda x)^{-2}}\right]^{\alpha},\ x>0,\alpha,\lambda>0. \tag{2}$$

Also, the corresponding reliability function is as follows

$$R\left(x\right)=1-F\left(x\right)=\left[1-e^{-(\lambda x)^{-2}}\right]^{\alpha},\ x>0,\alpha,\lambda>0. \tag{3}$$

The hazard and the cumulative hazard functions of this distribution are given, respectively, by

$$h\left(x\right)=\frac{f\left(x\right)}{1-F\left(x\right)}=\frac{2\alpha}{\lambda^2x^3}e^{-(\lambda x)^{-2}}\left[1-e^{-(\lambda x)^{-2}}\right]^{-1}, \tag{4}$$

and

$$H\left(x\right)=-\log[R\left(x\right)]=-\alpha\log\left[1-e^{-(\lambda x)^{-2}}\right], \tag{5}$$

where $x>0,\alpha,\lambda>0.$

This article is organized as follows. In section 2, the maximum likelihood estimators are provided. Section 3 discusses the Bayesian estimations for the parameters of the GIRD under squared error (SE) and linear-exponential (LINEX) loss functions. In section 4, applications to real data sets are carried out for illustrative purpose. Finally, concluding remarks are given in section 5.

II. MAXIMUM LIKELIHOOD ESTIMATION AND INFORMATION MATRIX

In order to derive the maximum likelihood estimators (MLEs) of α and λ, we first find the likelihood function $L(\underline{x})$ for the random variables $X_1, X_2, \ldots, X_n$ from the GIRD

$$L\left(\underline{x}\right)=\left(\frac{2\alpha}{\lambda^2}\right)^n\prod_{i=1}^{n}x_i^{-3}e^{-\sum_{i=1}^{n}(\lambda x_i)^{-2}}\prod_{i=1}^{n}[1-e^{-(\lambda x_i)^{-2}}]^{\alpha-1} \tag{6}$$

then the log-likelihood function is as follows

$$\log L=n\log(2\alpha)-2n\log\lambda-3\sum_{i=1}^{n}\log(x_i)$$
$$-\sum_{i=1}^{n}(\lambda x_i)^{-2}+(\alpha-1)\sum_{i=1}^{n}\log(1-e^{-(\lambda x_i)^{-2}}).$$

By differentiate the log-likelihood function with respect to α and λ, respectively, then by equating the resulting equations to zero, we get

$$\frac{d\log L}{d\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left(1-e^{-(\lambda x_i)^{-2}}\right)$$

$$\frac{n}{\hat{\alpha}}=-\sum_{i=1}^{n}\log\left(1-e^{-(\lambda x_i)^{-2}}\right)$$

thus

$$\hat{\alpha}=\frac{-n}{\sum_{i=1}^{n}\log\left(1-e^{-(\lambda x_i)^{-2}}\right)} \tag{7}$$

Similarly, for λ

$$\frac{-2n}{\lambda}+\sum_{i=1}^{n}\frac{2}{\lambda^3 x_i^2}-\frac{2(\alpha-1)}{\lambda^3}\sum_{i=1}^{n}\frac{e^{-(\lambda x_i)^{-2}}}{x_i^2[1-e^{-(\lambda x_i)^{-2}}]}=0$$

which can be simplified as

$$\frac{-2n}{\lambda}+\sum_{i=1}^{n}\frac{2}{\lambda^3 x_i^2}-\frac{2(\alpha-1)}{\lambda^3}\sum_{i=1}^{n}[x_i^2(e^{(\lambda x_i)^{-2}}-1)]^{-1}=0 \tag{8}$$

This equation will be solved numerically to obtain the estimate of λ.

By using Newton-Raphson method, this equation will be solved numerically to obtain the estimator of λ, say $\hat{\lambda}$. Hence, the estimator of α could be obtained by substituting $\hat{\lambda}$ from (8) in (7).

A. Fisher Information Matrix

Under standard regularity conditions (see, [14]) that are fulfilled for the proposed model whenever the parameters are in the interior of the parameter space, the observed information matrix $I(\boldsymbol{\theta})$ can be employed for interval estimation of the model parameters. Suppose X is a random variable with likelihood function $L(\underline{x})$, then the information matrix $I(\boldsymbol{\theta})$ is the following 2 ×2 symmetric matrix

$$I(\boldsymbol{\theta})=-E\begin{bmatrix}\dfrac{\partial^2\log L}{\partial\alpha^2} & \dfrac{\partial^2\log L}{\partial\alpha\,\partial\lambda}\\[2ex] \dfrac{\partial^2\log L}{\partial\lambda\partial\alpha} & \dfrac{\partial^2\log L}{\partial\lambda^2}\end{bmatrix}\quad,\ \boldsymbol{\theta}=(\alpha,\lambda)'$$

where

$$\frac{\partial^2\log L}{\partial\alpha^2}=\frac{-n}{\alpha^2},$$

$$\begin{aligned}\frac{\partial^2\log L}{\partial\lambda^2}&=\frac{2n}{\lambda^2}-\frac{6}{\lambda^4}\sum_{i=1}^{n}\frac{1}{x_i^2}\\ &+\frac{6(\alpha-1)}{\lambda^4}\sum_{i=1}^{n}x_i^{-2}\left[e^{(\lambda x_i)^{-2}}-1\right]^{-1}\\ &-\frac{4(\alpha-1)}{\lambda^6}\sum_{i=1}^{n}\frac{e^{-(\lambda x_i)^{-2}}}{x_i^4\left[1-e^{-(\lambda x_i)^{-2}}\right]^2}\times\left\{2e^{-(\lambda x_i)^{-2}}-1\right\},\end{aligned}$$

$$\frac{\partial^2\log L}{\partial\alpha\,\partial\lambda}=\frac{\partial^2\log L}{\partial\lambda\partial\alpha}=\frac{-2}{\lambda^3}\sum_{i=1}^{n}x_i^{-2}\left[e^{(\lambda x_i)^{-2}}-1\right]^{-1}.$$

To obtain expectation of the above expression is tedious job. Therefore, we use the observed Fisher information matrix which is given by

$$I(\hat{\boldsymbol{\theta}})=\begin{bmatrix}-\dfrac{\partial^2\log L}{\partial\alpha^2} & -\dfrac{\partial^2\log L}{\partial\alpha\,\partial\lambda}\\[2ex] -\dfrac{\partial^2\log L}{\partial\lambda\partial\alpha} & -\dfrac{\partial^2\log L}{\partial\lambda^2}\end{bmatrix}_{\alpha=\hat{\alpha},\lambda=\hat{\lambda}}$$

The asymptotic variance-covariance matrix of the MLEs is the inverse of $I(\hat{\boldsymbol{\theta}})$. After obtaining inverse matrix, we get variance of $\hat{\alpha}$ and variance of $\hat{\lambda}$. We use these terms to obtain confidence intervals for α and λ respectively.

B. Confidence Intervals

Assuming asymptotic normal distribution for the MLEs, confidence intervals (CIs) for α and λ are constructed. Let $\hat{\alpha}$ and $\hat{\lambda}$ are the MLEs of α and λ respectively. Let $\operatorname{var}(\hat{\alpha})$ and $\operatorname{var}(\hat{\lambda})$ is the estimated variances of $\hat{\alpha}$ and $\hat{\lambda}$ respectively. Therefore, 100(1 − ξ)% asymptotic CIs for α and λ are respectively given by

$$\left(\hat{\alpha}-z_{\xi/2}\sqrt{\operatorname{var}(\hat{\alpha})},\hat{\alpha}+z_{\xi/2}\sqrt{\operatorname{var}(\hat{\alpha})}\right),$$

and $\left(\hat{\lambda}-z_{\xi/2}\sqrt{\operatorname{var}(\hat{\lambda})},\hat{\lambda}+z_{\xi/2}\sqrt{\operatorname{var}(\hat{\lambda})}\right)$,

where $z_{\xi/2}$ is the upper $100(1-\xi)^{th}$ percentile of standard normal distribution.

III. BAYESIAN ESTIMATION

A. Case 1: Bayes Estimation when α and λ are Unknown

Suppose α and λ are unknown, independent and have the following prior distributions

$\lambda\,\square\,\mathit{InvertedGamma}(a,b),\ \alpha\,\square\,\mathit{Gamma}(c,d).$

Thus, the joint prior is given by

$$\pi(\lambda,\alpha)=\frac{b^a}{\Gamma(a)}\lambda^{-a-1}e^{-b\lambda^{-1}}\ \frac{d^c}{\Gamma(c)}\alpha^{c-1}e^{-d\alpha}, \tag{9}$$

where $\alpha,\lambda,a,b,c,d>0.$

Then, after multiplying (6) and (9), the joint posterior density function of α and λ given the data is given by

$$\begin{aligned}\pi^*(\lambda,\alpha/\underline{x})&=k\,\lambda^{-a-2n-1}\alpha^{n+c-1}\exp\left\{-b\lambda^{-1}\right\}\\ &\times\exp\left\{-\alpha\left(d-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right)\right\}\\ &\times\exp\left\{-\sum_{i=1}^{n}(\lambda x_i)^{-2}-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right\},\end{aligned} \tag{10}$$

where k is a normalizing constant and have the following form

$$\begin{aligned}k^{-1}&=\int_0^\infty\!\!\int_0^\infty\lambda^{-a-2n-1}\alpha^{n+c-1}\exp\left\{-b\lambda^{-1}\right\}\\ &\times\exp\left\{-\alpha\left(d-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right)\right\}\\ &\times\exp\left\{-\sum_{i=1}^{n}(\lambda x_i)^{-2}-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right\}d\lambda\,d\alpha.\end{aligned} \tag{11}$$

The Bayes estimators of a function of α and λ denoted by $\varphi(\alpha,\lambda)$ are obtained under two different types of loss functions. For computing the Bayes estimators, we use the standard Bayes and the importance sampling techniques.

i) Squared Error Loss Function

The Bayes estimators of $\varphi(\lambda,\alpha)$ under squared error (SE) loss function, using standard Bayes technique, denoted by $\hat{\varphi}_{SS}(\lambda,\alpha)$, is the posterior mean assuming that exists. It can be obtained as follows

$$\hat{\varphi}_{SS}(\lambda,\alpha) = k\int_0^\infty\int_0^\infty \varphi(\alpha,\lambda)\,\lambda^{-a-2n-1}\alpha^{n+c-1}\,\mathrm{e}^{-b\lambda^{-1}} \times\exp\left\{-\alpha\left(d-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right)\right\} \times\exp\left\{-\sum_{i=1}^{n}(\lambda x_i)^{-2}-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right\}d\lambda\,d\alpha, \quad (12)$$

where k^{-1} is defined in (11).

Next, the Bayes estimator using importance sampling technique can be derived by using the joint posterior density function (10) as follows.

The joint posterior density function of α and λ can be rewritten as

$$\pi^*(\lambda,\alpha/\underline{x}) \propto \frac{b^{a+2n}}{\Gamma(a+2n)}\lambda^{-a-2n-1}\,\mathrm{e}^{-b\lambda^{-1}}\frac{[\phi(\lambda)]^{n+c}}{\Gamma(n+c)}\alpha^{n+c-1} \times\exp\{-\alpha[\phi(\lambda)]\} \times\frac{\exp\left\{-\sum_{i=1}^{n}(\lambda x_i)^{-2}-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right\}}{[\phi(\lambda)]^{n+c}},$$

where $\phi(\lambda) = d-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right].$

Thus, the joint posterior density function of α and λ can be considered as

$$\pi^*(\lambda,\alpha/\underline{x}) \propto InvertedGamma\left(a+2n,b\right) \times Gamma\left(n+c,\phi(\lambda)\right)g(\lambda,\alpha/\underline{x}), \quad (13)$$

where

$$g(\lambda,\alpha/\underline{x}) = \frac{\exp\left\{-\sum_{i=1}^{n}(\lambda x_i)^{-2}-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right\}}{[\phi(\lambda)]^{n+c}}. \quad (14)$$

The Bayes estimators based on importance sampling technique of $\varphi(\lambda,\alpha)$ can be obtained by using the following algorithm.

Algorithm (1) for Importance Sampling

Step 1. Generate $\lambda_i \sim InvertedGamma(a+2n,b)$ and $\alpha_i \mid \lambda_i \sim Gamma(n+c,\phi(\lambda_i))$.

Step 2. Repeat step 1 to obtain (α_1, λ_1), (α_2, λ_2),…, (α_N, λ_N).

Step 3. Compute the value

$$\hat{\varphi}_{SIM}(\lambda,\alpha) = \frac{\sum_{j=1}^{N}\varphi(\lambda_j,\alpha_j)g(\lambda_j,\alpha_j/\underline{x})}{\sum_{j=1}^{N}g(\lambda_j,\alpha_j/\underline{x})}, \quad (15)$$

where

$$g(\lambda_j,\alpha_j/\underline{x}) = \frac{\exp\left\{-\sum_{i=1}^{n}(\lambda_j x_i)^{-2}-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda_j x_i)^{-2}}\right]\right\}}{[\phi(\lambda_j)]^{n+c}}, \quad (16)$$

and $\phi(\lambda_j) = d-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda_j x_i)^{-2}}\right].$

ii) Linear-exponential loss function

The Bayes estimators of $\varphi(\alpha,\lambda)$ under linear-exponential (LINEX) loss function, denoted by $\hat{\varphi}_{LS}(\alpha,\lambda)$, is given by

$$\hat{\varphi}_{LS}(\alpha,\lambda) = \frac{-1}{\tau}\log[k\int_0^\infty\int_0^\infty e^{-\tau\varphi(\alpha,\lambda)}\lambda^{-a-2n-1}\alpha^{n+c-1}\,\mathrm{e}^{-b\lambda^{-1}} \times\exp\{-\alpha[\phi(\lambda)]\} \times\exp\left\{-\sum_{i=1}^{n}(\lambda x_i)^{-2}-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right\}d\lambda\,d\alpha],$$

where k^{-1} is defined in (11).

According to the importance sampling technique, the approximate Bayes estimator under LINEX loss function, say $\hat{\varphi}_{LIM}(\alpha,\lambda)$ can be computed by applying the steps 1 and 2 in the algorithm (1), then the third step can be calculated as

Step 3. Compute the value

$$\hat{\varphi}_{LIM}(\lambda,\alpha) = -\frac{1}{\tau}\log\left[\frac{\sum_{j=1}^{N}e^{-\tau\varphi(\lambda_j,\alpha_j)}g(\lambda_j,\alpha_j/\underline{x})}{\sum_{j=1}^{N}g(\lambda_j,\alpha_j/\underline{x})}\right],$$

where $g(\lambda_j,\alpha_j/\underline{x})$ is defined in (16).

B. Case 2: Bayes Estimation when α is Unknown

Now, suppose λ is known and α is unknown and has the following prior distribution

$\alpha \sim Gamma(c,d).$

Therefore, the prior for α is given by

$$\pi(\alpha) = \frac{d^c}{\Gamma(c)}\alpha^{c-1}e^{-d\alpha}, \quad \alpha, c, d > 0. \quad (17)$$

Then, combining (6) and (17), the posterior density function of α given the data can be obtained as follows

$$\pi^*(\alpha/\underline{x}) = k\,\alpha^{n+c-1}\exp\left\{-\alpha\left(d-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right)\right\} \times\exp\left\{-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right\}, \quad (18)$$

where

$$k^{-1} = \int_0^\infty \alpha^{n+c-1}\exp\left\{-\alpha\left(d-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right)\right\} \times\exp\left\{-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right\}d\alpha. \quad (19)$$

The Bayes estimators of a function of α denoted by $\varphi(\alpha)$ are obtained under two different types of loss functions using the standard Bayes and the importance sampling techniques.

iii) Squared Error Loss Function

The Bayes estimator of $\varphi(\alpha)$ under SE loss function, denoted by $\hat{\varphi}_{SS}(\alpha)$, is the posterior mean assuming that exists. It is given by

$$\hat{\varphi}_{SS}(\alpha)=k\int_0^\infty \varphi(\alpha)\alpha^{n+c-1}\exp\left\{-\alpha\left(d-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right)\right\}$$
$$\times\exp\left\{-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right\}d\alpha, \qquad (20)$$

where k^{-1} is defined in (19).

By rewritten the joint posterior density function of *α,* we get

$$\pi^*(\alpha/\underline{x})\propto\frac{\left(d-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right)^{n+c}}{\Gamma(n+c)}\alpha^{n+c-1}$$
$$\times\exp\left\{-\alpha\left(d-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right)\right\}$$
$$\times\exp\left\{-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right\}.$$

then, the posterior density function of *α* can be considered as

$$\pi^*(\alpha/\underline{x})\propto Gamma\left(n+c,\,d-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right)$$
$$\times g(\alpha/\underline{x}),$$

where

$$g(\alpha/\underline{x})=1. \qquad (21)$$

The Bayes estimator based on importance sampling technique of $\varphi(\alpha)$ can be obtained by using the following algorithm.

Algorithm (2) for Importance Sampling

Step 1. Generate $\alpha_i \sim Gamma\left(n+c,\,d-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right).$

Step 2. Repeat step 1 to obtain $\alpha_1, \alpha_2, \ldots, \alpha_N$.

Step 3. Compute the value

$$\hat{\varphi}_{SIM}(\alpha)=\frac{\sum_{j=1}^{N}\varphi(\alpha_j)g(\alpha_j/\underline{x})}{\sum_{j=1}^{N}g(\alpha_j/\underline{x})}, \qquad (22)$$

where $g(\alpha_j/\underline{x})=1 \qquad (23)$

iv) Linear-exponential loss function

The Bayes estimator of $\varphi(\alpha)$ under LINEX loss function, denoted by $\hat{\varphi}_{LS}(\alpha)$, is given by

$$\hat{\varphi}_{LS}(\alpha)=-\frac{1}{\tau}\log[k\int_0^\infty e^{-\tau\varphi(\alpha)}\alpha^{n+c-1}$$
$$\times\exp\left\{-\alpha\left(d-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right)\right\}$$
$$\times\exp\left\{-\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right]\right\}d\alpha], \qquad (24)$$

where k^{-1} is defined in (19).

The approximate Bayes estimator under LINEX loss function, say $\hat{\varphi}_{LIM}(\alpha)$ can be computed by applying the steps 1 and 2 in the algorithm (2), then the third step can be calculated as

Step 3. Compute the value

$$\hat{\varphi}_{LIM}(\alpha)=-\frac{1}{\tau}\log\left[\frac{\sum_{j=1}^{N}e^{-\tau\varphi(\alpha_j)}g(\alpha_j/\underline{x})}{\sum_{j=1}^{N}g(\alpha_j/\underline{x})}\right],$$

where $g(\alpha_j/\underline{x})$ is defined in (23).

C. Case 3: Bayes Estimation when λ is Unknown

Suppose *α* is known and *λ* is unknown and has the following prior distribution

$\lambda \sim InvertedGamma(a,b).$

Thus, the prior for *λ* is given by

$$\pi(\lambda)=\frac{b^a}{\Gamma(a)}\lambda^{-a-1}e^{-b\lambda^{-1}},\quad \lambda,a,b>0. \qquad (25)$$

Then, combining (6) and (25), the posterior density function of *λ* given the data can be written as follows

$$\pi^*(\lambda/\underline{x})=k\,\lambda^{-a-2n-1}\exp\left[-b\lambda^{-1}-\zeta(\lambda)\right],$$

where

$$\zeta(\lambda)=\sum_{i=1}^{n}(\lambda x_i)^{-2}-(\alpha-1)\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i)^{-2}}\right], \qquad (26)$$

and *k* is a normalizing constant and can be written as

$$k^{-1}=\int_0^\infty\lambda^{-a-2n-1}\exp\left[-b\lambda^{-1}-\zeta(\lambda)\right]d\lambda. \qquad (27)$$

In the following, the Bayes estimators of a function of *λ*, denoted by $\varphi(\lambda)$, will be obtained under SE and LINEX loss functions using the standard Bayes and the importance sampling techniques.

i) Squared Error Loss Function

The Bayes estimator of $\varphi(\lambda)$ under SE loss function, denoted by $\hat{\varphi}_{SS}(\lambda)$, is the posterior mean assuming that exists. It is given by

$$\hat{\varphi}_{SS}(\lambda)=k\int_0^\infty\varphi_{SS}(\lambda)\lambda^{-a-2n-1}\exp\left[-b\lambda^{-1}-\zeta(\lambda)\right]d\lambda, \quad (28)$$

where $\zeta(\lambda)$ and k^{-1} are defined in (26) and (27), respectively.

Next, By rewritten the joint posterior density function of *λ,* we get

$$\pi^*(\lambda/\underline{x})\propto\frac{(b)^{a+2n}}{\Gamma(a+2n)}\lambda^{-a-2n-1}\exp\left[-b\lambda^{-1}-\zeta(\lambda)\right].$$

Then, the posterior density function of *λ* can be considered as

$$\pi^*(\lambda/\underline{x})\propto InvertedGamma(a+2n,\,b)\,g(\lambda/\underline{x}),$$

where

$$g(\lambda/\underline{x})=\exp\left[-\zeta(\lambda)\right]. \qquad (29)$$

The Bayes estimator based on importance sampling technique of $\varphi(\lambda)$ can be obtained by using the following algorithm.

Algorithm (3) for Importance Sampling

Step 1. Generate $\lambda_i \sim InvertedGamma(a+2n, b)$.

Step 2. Repeat step 1 to obtain $\lambda_1, \lambda_2, \ldots, \lambda_N$.

Step 3. Compute the value

$$\hat{\varphi}_{SIM}(\lambda) = \frac{\sum_{j=1}^{N} \varphi(\lambda_j) g(\lambda_j / \underline{x})}{\sum_{j=1}^{N} g(\lambda_j / \underline{x})}, \tag{30}$$

where

$$g(\lambda_j / \underline{x}) = \exp\left[-\zeta(\lambda_j)\right]. \tag{31}$$

ii) Linear-exponential loss function

The Bayes estimator of $\varphi(\lambda)$ under LINEX loss function, denoted by $\hat{\varphi}_{LS}(\lambda)$, is given by

$$\hat{\varphi}_{LS}(\lambda) = -\frac{1}{\tau}\log[k\int_0^\infty e^{-\tau\varphi(\lambda)}\lambda^{-a-2n-1}$$
$$\times \exp\left[-b\lambda^{-1} - \zeta(\lambda)\right] d\lambda],$$

where $\zeta(\lambda)$ and k^{-1} are defined in (26) and (27), respectively.

Next, by using the importance sampling technique, the approximate Bayes estimator under LINEX loss function, say $\hat{\varphi}_{LIM}(\lambda)$ can be computed by applying the steps 1 and 2 in the algorithm (3), then the third step can be calculated as

Step 3. Compute the value

$$\hat{\varphi}_{LIM}(\lambda) = -\frac{1}{\tau}\log\left[\frac{\sum_{j=1}^{N} e^{-\tau\varphi(\lambda_j)} g(\lambda_j / \underline{x})}{\sum_{j=1}^{N} g(\lambda_j / \underline{x})}\right], \tag{32}$$

where $g(\lambda_j / \underline{x})$ is defined in (31).

IV. APPLICATIONS

To illustrate the estimation techniques that were shown in the previous sections, we present two data sets. To test the goodness of fit, we consider five criteria such as negative log-likelihood, Kolmogorov- Smirnov (K-S) statistic, Akaike's information criterion (AIC), CAIC (corrected Akaike's information criterion) and Bayesian information criterion (BIC). AIC, CAIC and BIC are respectively given by

$\text{AIC} = 2k - 2\log(L)$,

$$\text{CAIC} = \text{AIC} + \frac{2k(k+1)}{n-k-1},$$

and $\text{BIC} = k\log(n) - 2\log(L)$,

where k is the number of parameters in the reliability model, L is the maximized value of the likelihood function for the estimated model and n is the number of the observations in the given data set.

The first data set

This data set is obtained from [16] and represents times between successive failures of air conditioning (AC) equipment in a Boeing 720 airplane and they are as follows: 502, 386, 326, 153, 74, 70, 59, 57, 48, 29, 29, 27, 26, 21, 12.

The maximum likelihood estimators of parameters, values of negative log-likelihood, AIC, CAIC, BIC and K-S values of the GIRD and IRD are presented in Table (I).

Table I: The MLEs of the models parameters, AIC, CAIC, BIC, negative log-likelihood, and K-S statistics of the successive failures of AC equipment.

Distribution	MLEs	AIC	CAIC	BIC	-2Log$(\hat{\theta})$	K- S
GIRD	$\hat{\alpha} = 0.4338$ $\hat{\lambda} = 0.0441$	172.36	173.36	173.78	168.36	0.1116
IRD	$\hat{\lambda} = 0.0325$	179.88	180.19	180.59	177.88	0.2815

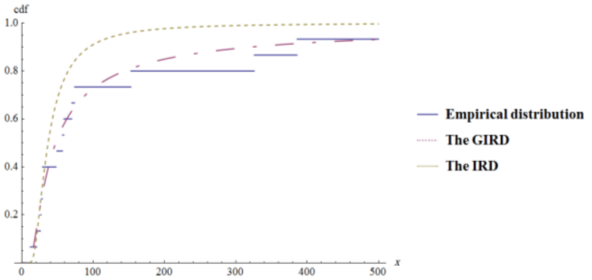

Fig.1. Plots of the GIRD and IRD with the Empirical of the successive failures of AC equipment..

According to Fig. 1 and to the criteria used for goodness of fit, the GIRD fits to the data set better than IRD. The ML and the Bayes estimators of α, λ, $R(x)$ and $H(x)$ are computed using these actual values { α=1.2, λ=0.032, x_0 =100, $R(x_0)$ = 0.05786 and $H(x_0)$ = 2.84968}. Also, the standard errors (std) for α and λ are obtained. The std of α is 0.03364, and the std of λ is 0.00250. The estimates are given in Tables (II) and (III).

As shown from the below tables, the ML estimates for all parameters are better than the Bayes estimates according to the two techniques used; importance sampling and standard Bayes.

Table II: ML and Bayes estimates using standard Bayes method of the successive failures of AC equipment.

n	parameters	MLE	BS	BL		
				$\tau = 2.5$	$\tau = 1$	$\tau = 0.001$
15	α	0.4338	0.0636	0.0633	0.0635	0.0636
	λ	0.04415	0.37451	0.31862	0.34499	0.3744
	$R(x)$	0.27258	0.65120	0.64462	0.64859	0.6512
	$H(x)$	1.29982	0.43528	0.41989	0.42892	0.4352

Table III: Bayes estimates using importance sampling method of the successive failures of AC equipment.

n	parameters	BS	BL		
			$\tau = 2.5$	$\tau = 1$	$\tau = 0.001$
15	α	0.0579	0.0579	0.0579	0.0579
	λ	0.5553	0.5541	0.5549	0.5553
	$R(x)$	0.6287	0.6268	0.6279	0.6287
	$H(x)$	0.4659	0.4608	0.4640	0.4659

The second data set

The data below show survival times (in months) of patients with Hodgkin's disease, heavy therapy, who were treated with nitrogen mustards, which is taken from [17].

1.05, 2.92, 3.61, 4.20, 4.49, 6.72, 7.31, 9.08, 9.11, 14.49, 16.85, 18.82, 26.59, 30.26, 41.34.

This data fits the GIRD better than the IRD, Table IV shows the criteria used for goodness of fit. Also, Fig. 2 shows that the curve of the GIRD fits the empirical data.

Table IV: The MLEs of the models parameters, AIC, CAIC, BIC, negative log-likelihood, and K-S statistics of the survival times (in months) of patients with Hodgkin's disease.

Distribution	MLEs	AIC	CAIC	BIC	$-2\text{Log}(\hat{\theta})$	K- S
GIRD	$\hat{\alpha} = 0.3457$ $\hat{\lambda} = 0.4487$	117.97	118.97	119.39	113.97	0.2092
IRD	$\hat{\lambda} = 0.2928$	132.49	132.80	133.20	130.49	0.4391

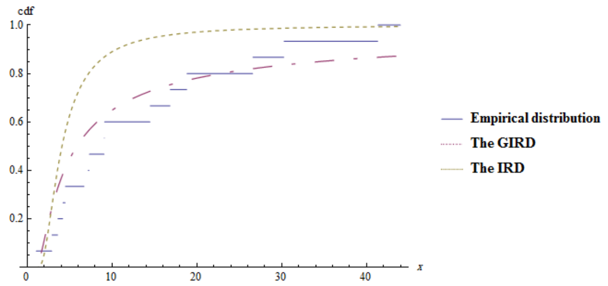

Fig.2. Plots of the GIRD and IRD with the Empirical of survival times (in months) of patients with Hodgkin's disease.

The ML and the Bayes estimators of α, λ, $R(x)$ and $H(x)$ are computed using these actual values { $\alpha = 1.2$, $\lambda = 0.032$, $x_0 = 20$, $R(x_0) = 0.89648$ and $H(x_0) = 0.10927$}. Also, the standard errors (std) for α and λ are obtained. The std of α is 0.02645, and the std of λ is 0.02779. The estimates are given in Tables (V) and (VI).

The ML estimates of α and λ are the best among all estimates as it is shown from Table V and VI. Whereas, the Bayes estimates are the best for $R(x)$ and $H(x)$.

Table V: ML and Bayes estimates using standard Bayes method of survival times (in months) of patients with Hodgkin's disease.

n	parameters	MLE	BS	BL		
				$\tau = 2.5$	$\tau = 1$	$\tau = 0.001$
15	α	0.3457	0.0896	0.0889	0.0893	0.0896
	λ	0.4487	0.8877	0.7564	0.8166	0.8876
	$R(x)$	0.2188	0.6137	0.6065	0.6109	0.6137
	$H(x)$	1.5195	0.4959	0.4769	0.4881	0.4959

Table VI: Bayes estimates using importance sampling method of survival times (in months) of patients with Hodgkin's disease.

n	parameters	BS	BL		
			$\tau = 2.5$	$\tau = 1$	$\tau = 0.001$
15	α	0.0811	0.0809	0.0811	0.0811
	λ	0.5467	0.5463	0.5466	0.5467
	$R(x)$	0.6792	0.6771	0.6783	0.67920
	$H(x)$	0.3886	0.3838	0.3868	0.3886

V. CONCLUDING REMARKS

In this article, we have studied the generalized inverse Rayleigh distribution as a new distribution proposed by [1]. Complete samples for the GIRD are considered for obtaining estimations for the unknown parameters, reliability and cumulative hazard functions of the model. This was treated for three cases; two cases assume one parameter is unknown, and the third case assumes all parameters of GIRD are unknown.

Two main estimation methods have been used; ML and Bayesian estimation. An asymptotic variance-covariance matrix and the approximate confidence intervals of the parameters have been derived. In addition, for Bayesian estimation, an inverted gamma distribution is assumed as a prior distribution for the scale parameter and a gamma distribution is assumed as a prior distribution for the shape parameter. SE and LINEX loss functions based on standard Bayes and importance sampling methods have been considered. The importance samples technique were performed according to the algorithm by [18]. However, the estimates cannot be obtained in explicit from, therefore, *Mathematica 9* program has been used to compute the estimates. The computations have been carried out using two real data sets. The ML estimates in both real data sets were the best for the two parameters of the GIRD. However, the Bayes estimates were the best for the reliability and hazard functions.

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