

Some Classes of Entire Sequence of Modals

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Abstract – In the past decades, modal analysis has become a major technology in the quest for determining, improving and optimizing dynamic characteristics of engineering structures. Not only has it been recognized in mechanical and aeronautical engineering, but modal analysis has also been discovered in profound applications for civil and building structures, space structures, transportation and nuclear problems [4]. Shortly, modal analysis relies on mathematics to establish theoretical models for adynamic system and to analyze data in various forms. Since modals are used in different branches of engineering in order to contribute to modal analysis, we have constructed some sequence space $G_{\mathcal{R}^2}(gI)$ of modal intervals. Also, we have given some new definitions and theorems about the sequence space $G_{\mathcal{R}^2}(gI)$ of modals.

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I. INTRODUCTION

Interval arithmetic was first suggested by Dwyer [2] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [9] in 1959 and Moore and Yang [10] 1962. Furthermore, Moore and others[11] have developed applications to differential equations.

Chiao in [5] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Sengönül and Eryilmaz [12] in 2010 studied bounded and convergent sequence space of interval numbers and showed that these spaces are complete metric space. Recently, Zararsiz and Sengönül[13] introduced null, bounded and convergent sequence space of modals.

Let us denote the set of all real valued closed interval by I , the set of positive integers by \mathbb{N} and the set of all real numbers by \mathcal{R} . Any element of I is called interval number and it is denoted by \hat{x} . That is $\hat{x} = \{x \in \mathcal{R} : x \leq \bar{x}\}$. An interval number \hat{x} is a closed subset of real numbers. Let \underline{x} and \bar{x} be respectively first and last points of the interval number \hat{x} . Therefore, when $\underline{x} > \bar{x}$, \hat{x} is not an interval number. But in modal analysis $[\bar{x}, \underline{x}]$ is a valid interval. A modal $\tilde{x} = \{[x, \bar{x}] : x, \bar{x} \in \mathcal{R}\}$ is defined by a pair of real numbers \bar{x}, x . Let us denote the set of all modals by gI . Let us suppose that $\tilde{x}, \tilde{y} \in gI$. Then the algebraic operations between \tilde{x} and \tilde{y} are defined in the Kaucher arithmetic, [7]. For a modal $\tilde{x} = [x, \bar{x}]$ dual operator is defined as $dual\tilde{x} = [\bar{x}, x]$. Thus, if $\tilde{x} \in gI$, then

$\tilde{x} - dual\tilde{x} = [0, 0] = \tilde{0}$, $dual\tilde{x} \in gI$. Let us suppose that $\tilde{x} \in gI$, then \tilde{x} is called symmetric modal if $\underline{x} = -\bar{x}$ or vice-versa.

The set of all modals gI is metric space defined as

$$d(\tilde{x}_1, \tilde{x}_2) = \max\{|\underline{x}_1 - \underline{x}_2|, |\bar{x}_1 - \bar{x}_2|\} \quad (1.1)$$

If $\tilde{x}, \tilde{y} \in gI$ and $\underline{x} \leq \bar{x}, \underline{y} \leq \bar{y}$ then the set gI is reduced ordinary set of interval numbers which is complete metric space with the metric d defined in (1.1)[7]. If we take $\tilde{x}_1 = [a, a]$ and $\tilde{x}_2 = [b, b]$, we obtain the usual metric of \mathcal{R} with $d(\tilde{x}_1, \tilde{x}_2) = |a - b|$, where $a, b \in \mathcal{R}$

Let f be a function from \mathbb{N} to gI which is defined by $k \rightarrow f(k) = \tilde{x}, \tilde{x} = (\tilde{x}_k)$. Then (\tilde{x}_k) is called sequence of modals. We will denote the set of all sequences of modals by $w(gI)$.

For two sequences of modals (\tilde{x}_k) and (\tilde{y}_k) , the addition, scalar product and multiplication are defined as follows $(\tilde{x}_k + \tilde{y}_k) = [\underline{x}_k + \underline{y}_k, \bar{x}_k + \bar{y}_k]$, $(\alpha\tilde{x}_k) = [\alpha\underline{x}_k, \alpha\bar{x}_k]$, $\alpha \in \mathcal{R}$, $(\tilde{x}_k \tilde{y}_k) = [\underline{x}_k \underline{y}_k, \bar{x}_k \bar{y}_k]$ respectively.

The set $w(gI)$ is a vector space since the vector space rules are clearly provided. The zero element of $w(gI)$ is the sequence $\tilde{\theta} = (\tilde{\theta}_k) = ([0, 0])$ all terms of which are zero interval. If $(\tilde{x}_k) \in w(gI)$ then inverse of (\tilde{x}_k) , according to addition, is $dual(\tilde{x}_k)$.

Let $\lambda(gI) \subset w(gI)$. If a sequence space contains a sequence (\tilde{e}_n) of modals with the property that for every $\tilde{u} \in \lambda(gI)$ there is a unique sequence of scalars (\tilde{t}_n) such that $\lim_n d(\tilde{u}, \tilde{t}_1\tilde{e}_1 + \dots + \tilde{t}_n\tilde{e}_n) \rightarrow \tilde{0}$ then (\tilde{e}_n) is called a Schauder modal basis for $\lambda(gI)$. The series $\sum_{k=1}^{\infty} \tilde{t}_k \tilde{e}_k$ which has the sum \tilde{u} is then called the expansion of \tilde{u} with respect to (\tilde{e}_n) , and we write $\tilde{u} = \sum_{k=1}^{\infty} \tilde{t}_k \tilde{e}_k$.

Let $\lambda(gI)$ and $\mu(gI)$ be linear space of modals. Then a function $\tilde{A} : \lambda(gI) \rightarrow \mu(gI)$ is called a linear transformation if and only if, for all $\tilde{u}_1, \tilde{u}_2 \in \lambda(gI)$ and all $\tilde{t}_1, \tilde{t}_2 \in gI$, $\tilde{A}(\tilde{t}_1\tilde{u}_1 + \tilde{t}_2\tilde{u}_2) = \tilde{t}_1\tilde{A}\tilde{u}_1 + \tilde{t}_2\tilde{A}\tilde{u}_2$.

Proposition 1.1. If (\tilde{x}_k) , (\tilde{y}_k) , (\tilde{r}_k) are sequences of symmetric modal, then the following equality holds:

$$(\tilde{x}_k)\{(\tilde{y}_k) - (\tilde{r}_k)\} = (\tilde{x}_k)(\tilde{y}_k) - (\tilde{x}_k)(\tilde{r}_k) \quad (1.2)$$

Definition 1.1. A sequence $\tilde{x} = (\tilde{x}_k) \in w(gI)$ of modals is said to be convergent to the modal \tilde{x}_0 if for each $\varepsilon > 0$ there exists a positive integer n_0 such that $d(\tilde{x}_k, \tilde{x}_0) < \varepsilon$ for all $k \geq n_0$ and we denote it by writing $\lim_k \tilde{x}_k = \tilde{x}_0$. Thus, $\lim_{k \rightarrow \infty} \tilde{x}_k = \tilde{x}_0 \Leftrightarrow \lim_{k \rightarrow \infty} \underline{x}_k = \underline{x}_0$ and $\lim_{k \rightarrow \infty} \bar{x}_k = \bar{x}_0$.

Definition 1.2. A sequence of modals, $\tilde{x} = (\tilde{x}_k) \in w(gI)$, is said to be modal fundamental sequence if for every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $d(\tilde{x}_k, \tilde{x}_n) < \varepsilon$ whenever $n, k > k_0$.

Definition 1.3. A sequence of modals $w(gI)$ is said to be solid if $\tilde{y} = (\tilde{y}_k) \in w(gI)$ whenever $\|\tilde{y}_k\| \leq \|\tilde{x}_k\|$ for all $k \in \mathbb{N}$ and $\tilde{x} = (\tilde{x}_k) \in w(gI)$.

Definition 1.4. A sequences of generalized intervals $w(gI)$ is said to be monotone if $w(gI)$ contains the canonical pre-image of all its step spaces.

Definition 1.5. A sequence of generalized intervals $w(gI)$ is said to be sequence algebra if $\tilde{x} \otimes \tilde{y} = (\tilde{x}_k \otimes \tilde{y}_k) \in w(gI)$ whenever $\tilde{x} = (\tilde{x}_k) \in w(gI)$ and $\tilde{y} = (\tilde{y}_k) \in w(gI)$.

We define convergent series, bounded series and p -absolute convergent series of sequences spaces of the symmetric modals which are denoted $cs(gI), bs(gI), l_p(gI)$ respectively, that is

$$cs(gI) = \left\{ \tilde{x} = (\tilde{x}_k) \in w(gI) : \lim_n \left(d \left(\sum_{k=1}^n \tilde{x}_k, \tilde{x}_n \right) \right) = \tilde{0} \right\},$$

$$bs(gI) = \left\{ \tilde{x} = (\tilde{x}_k) \in w(gI) : \sup_n \left(d \left(\sum_{k=1}^n \tilde{x}_k, \tilde{0} \right) \right) < \infty \right\},$$

$$l_p(gI) = \left\{ \tilde{x} = (\tilde{x}_k) \in w(gI) : \left(\sum_{k=1}^n [d(\tilde{x}_k, \tilde{0})]^p \right)^{1/p} < \infty, p \geq 1 \right\}$$

Clearly we see that the spaces $cs(gI), bs(gI)$ and $l_p(gI)$ are sub vector spaces in accordance with scalar product and addition on $w(gI)$ which are metric spaces.

We define the entire sequence spaces of symmetric modals which are denoted by $\Gamma(gI)$.

$$\Gamma(gI) = \left\{ \tilde{x} = (\tilde{x}_k) \in w(gI) : \lim_k (D(\tilde{x}_k, \tilde{0})) = \tilde{0} \right\},$$

where $D(\tilde{x}_k, \tilde{y}_k) = \max \left\{ \left| \underline{x}_k - \underline{y}_k \right|^{1/k}, \left| \bar{x}_k - \bar{y}_k \right|^{1/k} \right\}$

The function \tilde{d} defined by

$$\tilde{d}(\tilde{x}_k, \tilde{y}_k) = \sup_k \max \left\{ \left| \underline{x}_k - \underline{y}_k \right|^{1/k}, \left| \bar{x}_k - \bar{y}_k \right|^{1/k} \right\} \quad (1.3)$$

$$= \sup_k D(\tilde{x}_k, \tilde{y}_k)$$

which satisfies the metric space axioms.

Throughout this paper, let $\lambda = (\lambda_k)$ be a fined sequence of positive real number such that $\frac{\lambda_{k+1}}{\lambda_k} \rightarrow 1$ as $k \rightarrow \infty$ and

$\lambda_k \neq 1$ for all k . The space $G_{\lambda^2}(gI)$ is defined by

$$G_{\lambda^2}(gI) = \{ \tilde{u} = (\tilde{u}_k) \in w(gI) : \sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{u}_k, 0)]^2 < \infty \} \quad (1.4)$$

Norm of $G_{\lambda^2}(gI)$ defined by

$$\| \tilde{u}_k \|_{G_{\lambda^2}} = \left\{ \sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{u}_k, 0)]^2 \right\}^{1/2} \quad (1.5)$$

II. MAIN RESULTS

Theorem 2.1. The sequence space $G_{\lambda^2}(gI)$ is a complete metric space with norm defined by (1.5)

Proof: We will show that $G_{\lambda^2}(gI)$ is complete space with norm (1.5)

Let us suppose that $\tilde{u} = (\tilde{u}_k)$ is a fundamental sequence in $G_{\lambda^2}(gI)$ where $(\tilde{u}_k) = (\tilde{u}_0^i, \tilde{u}_1^i, \tilde{u}_2^i, \dots, \tilde{u}_k^i, \dots)$. Since the norm on $G_{\lambda^2}(gI)$ is given in(1.5) and (\tilde{u}_k) is a fundamental sequence, then for every $\varepsilon > 0$, there exists

$$k_0 \in \mathbb{N} \text{ such that } \| \tilde{u}_k^i - \tilde{u}_k^j \| = \left\{ \sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{u}_k^i - \tilde{u}_k^j, 0)]^2 \right\}^{1/2} < \varepsilon$$

for all $i, j > k_0$

In this case, We can write

$$\left\{ \sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{u}_k^i - \tilde{u}_k^j, 0)]^2 \right\} < \varepsilon^2$$

$$\lambda_k^2 [d(\tilde{u}_k^i - \tilde{u}_k^j, 0)]^2 < \varepsilon^2$$

$$[d(\tilde{u}_k^i - \tilde{u}_k^j, 0)]^2 < \varepsilon^2 / \lambda_k^2$$

$$[d(\tilde{u}_k^i - \tilde{u}_k^j, 0)] < \varepsilon / \lambda_k < \varepsilon \text{ for all } i, j > k_0$$

$$\max \{ d(\underline{u}_k^i - \underline{u}_k^j, 0), d(\bar{u}_k^i - \bar{u}_k^j, 0) \} < \varepsilon \text{ for all } i, j > k_0$$

This shows that, for every $k_0 \in \mathbb{N}$ the sequence (\tilde{u}_k^i) is fundamental sequence in gI . Since gI is a Banach space, the sequence (\tilde{u}_k^i) is convergent.

Now let $\lim_{i \rightarrow \infty} \tilde{u}_k^i = \tilde{u}_k, k \in \mathbb{N}$

If we consider to the inequality $\| \tilde{u}_k^i - \tilde{u}_k^j \| < \varepsilon$ whenever $i, j > k_0$

Then we have $\lim_{i \rightarrow \infty} \| \tilde{u}_k^i - \tilde{u}_k^j \| < \varepsilon$

$$\left\| \lim_{i \rightarrow \infty} \tilde{u}_k^i - \tilde{u}_k^j \right\| < \varepsilon$$

Since $\lim_{i \rightarrow \infty} \tilde{u}_k^i = \tilde{u}_k, \| \tilde{u}_k - \tilde{u}_k^j \| < \varepsilon$

That is, the sequence of generalized intervals \tilde{u}_k^j is convergent to \tilde{u}_k .

On the other hand,

$$\| \tilde{u}_k - [\tilde{u}_k^i - \tilde{u}_k^i] \| = \sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{u}_k - (\tilde{u}_k^i - \tilde{u}_k^i), 0)]^2$$

$$= \sum_{k=1}^{\infty} \lambda_k^2 \left\{ \max \left[\left| \underline{u}_k - (\underline{u}_k^i - \underline{u}_k^i) \right|, \left| \bar{u}_k - (\bar{u}_k^i - \bar{u}_k^i) \right| \right] \right\}^2$$

$$\leq \sum_{k=1}^{\infty} \lambda_k^2 \left\{ \max \left(\left| \underline{u}_k - \underline{u}_k^i \right|, \left| \bar{u}_k - \bar{u}_k^i \right| \right) \right\}^2 + \sum_{k=1}^{\infty} \lambda_k^2 \left\{ \max \left(\left| \underline{u}_k^i \right|, \left| \bar{u}_k^i \right| \right) \right\}^2$$

$< \varepsilon + \infty < \infty$

Therefore, $\tilde{u}_k \in G_{\lambda^2}(gI)$ which completes the proof.

Theorem 2.2. $G_{\lambda^2}(gI)$ is a subset of $\Gamma(gI)$

Proof: Let $\tilde{u} \in G_{\lambda^2}(gI)$, then

$$\sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{u}_k, 0)]^2 < \infty \quad (2.1)$$

$$\text{where } \frac{\lambda_{k+1}}{\lambda_k} \rightarrow 1 \text{ as } k \rightarrow \infty \text{ and } \lambda_k \neq 1 \text{ for all } k. \quad (2.2)$$

We claim that $\lim_{k \rightarrow \infty} [d(\tilde{u}_k, 0)]^{1/k} = 0$

From equation (2.1), $\lambda_k^2 [d(\tilde{u}_k, 0)]^2 < \varepsilon^{2k}$ for all $k \in \mathbb{N}$

$$\begin{aligned} [d(\tilde{u}_k, 0)]^2 &< \varepsilon^{2k} / \lambda_k^2 \\ d(\tilde{u}_k, 0) &< \varepsilon^k / \lambda_k \\ [d(\tilde{u}_k, 0)]^{1/k} &< \varepsilon / \lambda_k^{1/k} < \varepsilon_1 \text{ from (2.2)} \end{aligned}$$

Hence $[d(\tilde{u}_k, 0)]^{1/k} \rightarrow 0$ as $k \rightarrow \infty$ and so $\tilde{u} \in \Gamma(gI)$

Consequently $G_{\lambda^2}(gI)$ is a subset of $\Gamma(gI)$.

Theorem 2.3. If $G_{\lambda^2}(gI)$ and $G_{\mu^2}(gI)$ are two sequences of interval numbers, then $G_{\lambda^2}(gI) = G_{\mu^2}(gI)$

if and only if $k_1 \leq \frac{\lambda_k}{\mu_k} \leq k_2$ where k_1 and k_2 are constants.

Proof. The sufficiency of the condition

$$k_1 \leq \frac{\lambda_k}{\mu_k} \leq k_2 \quad (2.3)$$

If $\lambda_k \leq k_2 \mu_k$ then, $\lambda_k^2 [d(\tilde{u}_k, 0)]^2 \leq k_2^2 \mu_k^2 [d(\tilde{u}_k, 0)]^2$

If $\tilde{u} \in G_{\mu^2}(gI)$, then $\sum_{k=1}^{\infty} \mu_k^2 [d(\tilde{u}_k, 0)]^2 < \infty$

Therefore, $\sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{u}_k, 0)]^2 \leq k_2^2 \sum_{k=1}^{\infty} \mu_k^2 [d(\tilde{u}_k, 0)]^2 < \infty$

This implies $\tilde{u} \in G_{\lambda^2}(gI)$

$$\text{Hence } G_{\mu^2}(gI) \subset G_{\lambda^2}(gI) \quad (2.4)$$

$$\text{Similarly, if } k_1 \mu_k \leq \lambda_k \text{ then } G_{\lambda^2}(gI) \subset G_{\mu^2}(gI) \quad (2.5)$$

From (2.4) and (2.5), $G_{\lambda^2}(gI) = G_{\mu^2}(gI)$

To prove the necessity of the condition, let us suppose that the condition is not satisfied.

First consider the right hand side inequality of (2.3). Let $\frac{\lambda_k}{\mu_k} \rightarrow \infty$ as $k \rightarrow \infty$. Then, it has a subsequence

$\frac{\lambda_{k_n}}{\mu_{k_n}} \rightarrow \infty$ as $k_n \rightarrow \infty$ in such a manner that $\frac{\lambda_{k_n}}{\mu_{k_n}} > n$ for the

values $n=1,2,3,\dots$ and $k_1 < k_2 < \dots$

Now we shall define a sequence (\tilde{u}_k) as follows

$$\tilde{u}_k = \begin{cases} \left[\frac{1}{n \mu_k}, 0 \right] & \text{when } k = k_n \\ [0, 0] & \text{when } k \neq k_n \end{cases}$$

Then $\sum_{k=1}^{\infty} \mu_k^2 [d(\tilde{u}_k, 0)]^2 = \sum_{n=1}^{\infty} \mu_{k_n}^2 [d(\tilde{u}_{k_n}, 0)]^2$

$$= \sum_{n=1}^{\infty} \frac{\mu_{k_n}^2}{n^2 \mu_{k_n}^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Therefore $(\tilde{u}_k) \in G_{\mu^2}(gI)$ (2.6)

$$\begin{aligned} \text{But } \sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{u}_k, 0)]^2 &= \sum_{n=1}^{\infty} \lambda_{k_n}^2 [d(\tilde{u}_{k_n}, 0)]^2 \\ &> \sum_{n=1}^{\infty} n^2 \mu_{k_n}^2 [d(\tilde{u}_{k_n}, 0)]^2 = \sum_{n=1}^{\infty} \frac{n^2 \mu_{k_n}^2}{n^2 \mu_{k_n}^2} = \sum_{n=1}^{\infty} 1 > \infty \end{aligned}$$

Therefore $(\tilde{u}_k) \in G_{\lambda^2}(gI)$ (2.7)

(2.6) and (2.7) contradicts (2.4)

Similarly, If the left hand side inequity (2.3) is not satisfied then we can contradict (2.5) by constructing a sequence of the above type.

Hence, $k_1 \leq \frac{\lambda_k}{\mu_k} \leq k_2$ is necessary and sufficient in order

that $G_{\lambda^2}(gI) = G_{\mu^2}(gI)$.

Theorem 2.4. $G_{\lambda^2}(gI)$ is AK space.

Proof. For each $(\tilde{u}_k) \in G_{\lambda^2}(gI)$

$$\|\tilde{u}^{[n]} - \tilde{u}\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

Hence $G_{\lambda^2}(gI)$ has AK.

Theorem 2.5. $G_{\lambda^2}(gI)$ has AB property.

Proof. It is enough to show that $G_{\lambda^2}(gI)$ has monotone

norm. Indeed for $n < m$ and for every $(\tilde{u}_k) \in G_{\lambda^2}(gI)$

We have,

$$\|\tilde{u}^{[n]}\|^2 = \sum_{k=1}^n \lambda_k^2 [d(\tilde{u}_k, 0)]^2 < \sum_{k=1}^m \lambda_k^2 [d(\tilde{u}_k, 0)]^2 = \|\tilde{u}^{[m]}\|^2$$

Therefore, $\|\tilde{u}^{[n]}\|^2 < \|\tilde{u}^{[m]}\|^2$

Also $\{\|\tilde{u}^{[n]}\|, n=1,2,\dots\}$ is a monotonically increasing sequence of modal intervals bounded above by $\|\tilde{u}\|_{G_{\lambda^2}(gI)}$.

$$\text{Hence } \|\tilde{u}\|_{G_{\lambda^2}(gI)} = \lim_{n \rightarrow \infty} \|\tilde{u}^{[n]}\| = \sup_n \{\|\tilde{u}^{[n]}\|, n=1,2,\dots\}$$

Thus $G_{\lambda^2}(gI)$ has monotone norm.

Theorem 2.6. The space $G_{\lambda^2}(gI)$ is solid

Proof. Let (\tilde{u}_k) and (\tilde{v}_k) be two sequences such that

$(\tilde{u}_k) \in G_{\lambda^2}(gI)$ and

$$d(\tilde{v}_k, 0) \leq d(\tilde{u}_k, 0) \text{ for all } k \in \mathbb{N}$$

Since $(\tilde{u}_k) \in G_{\lambda^2}(gI)$, we have $\sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{u}_k, 0)]^2 < \infty$

Also we have, $\lambda_k^2 [d(\tilde{v}_k, 0)]^2 \leq \lambda_k^2 [d(\tilde{u}_k, 0)]^2$

$$\sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{v}_k, 0)]^2 \leq \sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{u}_k, 0)]^2 < \infty$$

So $(\tilde{v}_k) \in G_{\lambda^2}(gI)$ Therefore $G_{\lambda^2}(gI)$ is solid.

Theorem 2.7. The space $G_{\lambda^2}(gI)$ is symmetric.

Proof. Let (\tilde{u}_k) be a sequence in $G_{\lambda^2}(gI)$

Then $\sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{u}_k, 0)]^2 < \infty$

For $\varepsilon > 0$, there exists $k = k_0(\varepsilon)$ such that

$$\sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{u}_k, 0)]^2 - \sum_{k \leq k_0} \lambda_k^2 [d(\tilde{u}_k, 0)]^2 < \varepsilon$$

Let (\tilde{v}_k) be a rearrangement of (\tilde{u}_k) and k_1 be such that $\{(\tilde{u}_k) : k \leq k_0\} \subseteq \{(\tilde{v}_k) : k \leq k_0\}$

$$\text{Then } \sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{v}_k, 0)]^2 - \sum_{k \leq k_1} \lambda_k^2 [d(\tilde{v}_k, 0)]^2 < \varepsilon$$

and so $\sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{v}_k, 0)]^2 < \infty$

Hence $(\tilde{v}_k) \in G_{\lambda^2}(gI)$

$G_{\lambda^2}(gI)$ is symmetric.

Theorem 2.8. The space $G_{\lambda^2}(gI)$ is sequence Algebra.

Proof. We consider the space $G_{\lambda^2}(gI)$.

Let (\tilde{u}_k) and (\tilde{v}_k) be two sequences in $G_{\lambda^2}(gI)$ and $0 < \varepsilon < 1$. Then the result follows from the following inclusion relation

$$\{k \in \mathbb{N} : d(\tilde{u}_k \otimes \tilde{v}_k, 0)\} \supseteq \{k \in \mathbb{N} : d(\tilde{u}_k, 0)\} \cap \{k \in \mathbb{N} : d(\tilde{v}_k, 0)\}$$

Theorem 2.9.

$$G_{\lambda^2}^{\alpha}(gI) = G_{\lambda^2}^{\beta}(gI) = G_{\lambda^2}^{\gamma}(gI) = G_{\lambda^2}^f(gI) = H(gI)$$

$$\text{where } H(gI) = \left\{ \tilde{a} = (\tilde{a}_k) : \sum \frac{[d(\tilde{a}_k, 0)]^2}{\lambda_k^2} \text{ converges} \right\}$$

Proof. For $(\tilde{u}) \in G_{\lambda^2}(gI)$

$$\|\tilde{u}^{[n]} - \tilde{u}\|^2 = \left[\sum_{k=n+1}^{\infty} \lambda_k^2 [d(\tilde{u}_k, 0)]^2 \right]^{1/2} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Let $\tilde{a} = (\tilde{a}_k)$ be in $H(gI)$. For every $(\tilde{u}) \in G_{\lambda^2}(gI)$

$$d\left(\sum_{k=1}^{\infty} \tilde{a}_k \tilde{u}_k, 0\right) \leq \sum_{k=1}^{\infty} d(\tilde{a}_k \tilde{u}_k, 0) \leq \left[\sum_{k=1}^{\infty} \frac{[d(\tilde{a}_k, 0)]^2}{\lambda_k^2} \right]^{1/2} \left[\sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{u}_k, 0)]^2 \right]^{1/2}$$

Therefore $\sum_{k=1}^{\infty} \tilde{a}_k \tilde{u}_k$ converges for each $(\tilde{u}) \in G_{\lambda^2}(gI)$

$$\text{Consequently } H(gI) \subset G_{\lambda^2}^{\beta}(gI) \quad (2.8)$$

Let $\tilde{a} = (\tilde{a}_k)$ be any element $G_{\lambda^2}^{\beta}(gI)$

Then $\sum_{k=1}^{\infty} \tilde{a}_k \tilde{u}_k$ converges for each $(\tilde{u}) \in G_{\lambda^2}(gI)$

Define $\tilde{f}_n(\tilde{u}_k) = \sum_{k=1}^n \tilde{a}_k \tilde{u}_k, n = 1, 2, \dots$ for all $(\tilde{u}) \in G_{\lambda^2}(gI)$

Then

$$\begin{aligned} d(\tilde{f}_n(\tilde{u}_k), 0) &\leq \left[\sum_{k=1}^n \frac{[d(\tilde{a}_k, 0)]^2}{\lambda_k^2} \right]^{1/2} \left[\sum_{k=1}^n \lambda_k^2 [d(\tilde{u}_k, 0)]^2 \right]^{1/2} \\ d\left(\sum_{k=1}^n \tilde{a}_k \tilde{u}_k, 0\right) &\leq \left[\sum_{k=1}^n \frac{[d(\tilde{a}_k, 0)]^2}{\lambda_k^2} \right]^{1/2} M \end{aligned} \quad (2.9)$$

where $\sup \left\{ \left[\sum_{k=1}^n \lambda_k^2 [d(\tilde{u}_k, 0)]^2 \right]^{1/2}, n = 1, 2, \dots \right\} = M > 0$

Take any positive integer n, Define $\tilde{u} = (\tilde{u}_k)$ by

$$\tilde{u}_k = \begin{cases} \tilde{a}_k \operatorname{sgn} \tilde{a}_k / \lambda_k^2 & \text{for } 1 \leq k \leq n \\ \tilde{0} & \text{for } k > n \end{cases} \quad (2.10)$$

Then $(\tilde{u}_k) \in G_{\lambda^2}(gI)$

From (2.9),

$$d\left(\sum_{k=1}^n \tilde{a}_k \tilde{u}_k, 0\right) \leq \sum_{k=1}^n d(\tilde{a}_k, 0) d(\tilde{u}_k, 0) \leq \left[\sum_{k=1}^n \frac{[d(\tilde{a}_k, 0)]^2}{\lambda_k^2} \right]^{1/2} M$$

Using (2.10) in above inequality, we get

$$\sum_{k=1}^n \frac{[d(\tilde{a}_k, 0)]^2}{\lambda_k^2} \leq \left[\sum_{k=1}^n \frac{[d(\tilde{a}_k, 0)]^2}{\lambda_k^2} \right]^{1/2} M$$

$$\left[\sum_{k=1}^n \frac{[d(\tilde{a}_k, 0)]^2}{\lambda_k^2} \right]^{1/2} \leq M$$

$$\text{Let } n \rightarrow \infty, \left[\sum_{k=1}^{\infty} \frac{[d(\tilde{a}_k, 0)]^2}{\lambda_k^2} \right]^{1/2} \leq M$$

Hence $\tilde{a} = (\tilde{a}_k)$ belongs to $H(gI)$.

$$\text{Consequently, } G_{\lambda^2}^{\beta}(gI) \subset H(gI) \quad (2.11)$$

Using (2.8) and (2.11), $G_{\lambda^2}^{\beta}(gI) = H(gI)$

Hence from theorem (2.4) and (2.6),

$$G_{\lambda^2}^{\alpha}(gI) = G_{\lambda^2}^{\beta}(gI) = G_{\lambda^2}^{\gamma}(gI) = G_{\lambda^2}^f(gI) = H(gI)$$

III. MATRIX TRANSFORMATION INVOLVING $G_{\lambda^2}(gI)$

Theorem 3.1. $\tilde{A} \in (G_{\lambda^2}(gI) : c_0(gI))$ if and only if

$$(i) \sup_n \left[\sum_{k=1}^{\infty} \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} \right]^{1/2} \leq M < \infty$$

for some constant $M > 0$ and

(ii) $\lim_{n \rightarrow \infty} \tilde{a}_{nk} = \tilde{0}$ for each fixed k.

Proof. Sufficiency:

Let $(\tilde{u}) \in G_{\lambda^2}(gI)$ then $\sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{u}_k, 0)]^2 < \infty$

where $\frac{\lambda_{k+1}}{\lambda_k} \rightarrow 1$ as $k \rightarrow \infty$ and $\lambda_k \neq 1$ for all k.

Choose fixed integers n_0, N_1 and $M > 0$

such that $\sum_{k=n_0+1}^{\infty} \lambda_k^2 [d(\tilde{u}_k, 0)]^2 < \frac{\varepsilon^2}{4M^2}$ and $\lambda_k < N_1^k$

$$\begin{aligned} \text{Then } d(\tilde{A}_n(\tilde{u}_k), 0) &= d\left[\sum_{k=1}^{\infty} \tilde{a}_{nk} \tilde{u}_k, 0\right] \leq \sum_{k=1}^{\infty} d(\tilde{a}_{nk} \tilde{u}_k, 0) \\ &= \sum_{k=1}^{n_0} d(\tilde{a}_{nk} \tilde{u}_k, 0) + \sum_{k=n_0+1}^{\infty} d(\tilde{a}_{nk} \tilde{u}_k, 0) \end{aligned}$$

Now

$$d(\tilde{A}_n(\tilde{u}_k), 0) \leq \left[\sum_{k=1}^{n_0} \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} \right]^{1/2} \left[\sum_{k=1}^{n_0} \lambda_k^2 d(\tilde{u}_k, 0)^2 \right]^{1/2} + \left[\sum_{k=n_0+1}^{\infty} \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} \right]^{1/2} \left[\sum_{k=n_0+1}^{\infty} \lambda_k^2 d(\tilde{u}_k, 0)^2 \right]^{1/2} \quad (3.1)$$

By (i), $\sup_n \left[\sum_{k=1}^{\infty} \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} \right]^{1/2} \leq M$ Also by (ii)

$$d(\tilde{a}_{nk}, 0) < \frac{\varepsilon}{2Q} \frac{1}{\sqrt{n_0}} N_1^k \quad (3.2)$$

Where $Q = \left[\sum_{k=1}^{\infty} \lambda_k^2 d(\tilde{u}_k, 0)^2 \right]^{1/2}$

From (3.1) and (3.2), we have $d(\tilde{A}_n(\tilde{u}_k), 0) < \varepsilon$

Therefore $\tilde{A}\tilde{u} \in \tilde{c}_0(gI)$

Necessity:

Clearly the sequence $(\delta^k) = (0, 0, \dots, 1, 0, \dots) = \tilde{x}$ where 1 in the k^{th} place belongs to $G_{\lambda^2}(gI)$. Hence $\tilde{A}_n(\tilde{u}_k) \in \tilde{c}_0(gI)$. But $d(\tilde{A}_n(\tilde{u}_k), 0) = [d(\tilde{a}_{nk}, 0)]$

Thus $\tilde{a}_{nk} \in \tilde{c}_0$ for each fixed k. Therefore $\lim_{n \rightarrow \infty} \tilde{a}_{nk} = \tilde{0}$ for each fixed k. This proves (i).

We know that $G_{\lambda^2}(gI)$ is a Banach space with norm $\|\tilde{u}\| = \left[\sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{a}_{nk}, 0)]^2 \right]^{1/2}$ and $\tilde{c}_0(gI)$ is a Banach space with norm $\|\tilde{u}\| = \sup_k [d(\tilde{u}_k, 0)], k = 1, 2, \dots$

for $(\tilde{u}) \in G_{\lambda^2}(gI)$ define $\tilde{A}_n(\tilde{u}) = \sum_{k=1}^{\infty} \tilde{a}_{nk} \tilde{u}_k$

Then $\tilde{A}_n(\tilde{u})$ is a sequence of continues linear functions on $G_{\lambda^2}(gI)$ such that $d(\tilde{A}_n(\tilde{u}_k), 0)$ is bounded for every $(\tilde{u}) \in G_{\lambda^2}(gI)$. Therefore by uniform boundedness principle, the sequence $\tilde{A}_n(\tilde{u})$ is bounded.

For $(\tilde{u}) \in G_{\lambda^2}(gI)$, $\tilde{A}\tilde{u} \in \tilde{c}_0(gI)$

Therefore $\tilde{A}_n(\tilde{u})$ converges for every $(\tilde{u}) \in G_{\lambda^2}(gI)$

By Banach steinhaus theorem, $\sup_k \|\tilde{A}_n\| < \infty \quad (3.3)$

Now, $d(\tilde{A}_n(\tilde{u}_k), 0) = d\left[\sum_{k=1}^{\infty} \tilde{a}_{nk} \tilde{u}_k, 0\right] \leq \sum_{k=1}^{\infty} d(\tilde{a}_{nk} \tilde{u}_k, 0)$

$$\leq \left[\sum_{k=1}^{\infty} \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} \right]^{1/2} \left[\sum_{k=1}^{\infty} \lambda_k^2 d(\tilde{u}_k, 0)^2 \right]^{1/2}$$

$$= \left[\sum_{k=1}^{\infty} \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} \right]^{1/2} \|\tilde{u}\|$$

ie, $\|\tilde{A}_n\| = \left[\sum_{k=1}^{\infty} \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} \right]^{1/2} \quad (3.4)$

for given $\varepsilon > 0$, choose N such that

$$\left[\sum_{k=1}^{\infty} \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} \right]^{1/2} - \varepsilon < \left[\sum_{k=1}^N \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} \right]^{1/2} \quad (3.5)$$

We may choose N such that $\sum_{k=1}^N \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} > 1$

Let $Q = \sum_{k=1}^N \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2}$

Construct a sequence \tilde{u} as follows

$$\tilde{u}_k = \begin{cases} \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} & 1 \leq k \leq N \\ [0, 0] & \text{for } k > N \end{cases}$$

Then $(\tilde{u}) \in G_{\lambda^2}(gI)$ and $\|\tilde{u}\| = \left[\sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{u}_k, 0)]^2 \right]^{1/2} = 1$

Now,

$$d(\tilde{A}_n(\tilde{u}_k), 0) = d\left[\sum_{k=1}^{\infty} \tilde{a}_{nk} \tilde{u}_k, 0\right] = \sqrt{Q} = \left[\sum_{k=1}^N \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} \right]^{1/2}$$

By $\left[\sum_{k=1}^N \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} \right]^{1/2} = d(\tilde{A}_n(\tilde{u}_k), 0)$

$$\left[\sum_{k=1}^N \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} \right]^{1/2} = \|\tilde{A}_n\|$$

By (3.5) $\left[\sum_{k=1}^{\infty} \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} \right]^{1/2} - \varepsilon < \|\tilde{A}_n\|$

$$\left[\sum_{k=1}^{\infty} \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} \right]^{1/2} < \|\tilde{A}_n\| \quad (3.6)$$

Using (3.6) and (3.4), we get $\|\tilde{A}_n\| = \left[\sum_{k=1}^{\infty} \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} \right]^{1/2}$

By (3.3) $\sup_n \left[\sum_{k=1}^{\infty} \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} \right]^{1/2} < \infty$

Thus (ii) holds.

Theorem 3.2. $\tilde{A} \in (G_{\lambda^2}(gI) : c(gI))$ if and only if

(i) $\sup \left[\left(\sum_{k=1}^{\infty} \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} \right)^{1/2} \right]_{n=1, 2, \dots} \leq M < \infty$ for some constant $M > 0$

(ii) $\lim_{n \rightarrow \infty} [d(\tilde{a}_{nk}, 0)] = d(\tilde{a}_k, 0)$ exists for each fixed k.

Theorem 3.3. $\tilde{A} \in (G_{\lambda^2}(gI) : l_{\infty}(gI))$ if and only if

$\sup \left[\left(\sum_{k=1}^{\infty} \frac{[d(\tilde{a}_{nk}, 0)]^2}{\lambda_k^2} \right) \right]_{n=1, 2, \dots} \leq M < \infty$ for some constant $M > 0$.

Theorem 3.4. $\tilde{A} \in (G_{\lambda^2}(gI) : l(gI))$ if and only if

$\sup \left[\left(\sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{a}_{nk}, 0)]^2 \right) \right]_{n=1, 2, \dots} \leq M < \infty$ for some constant $M > 0$

Proof. The determining set in $l(gI)$ is $\tilde{E} = \{\tilde{\delta}^1, \tilde{\delta}^2, \dots\}$.

Let $A(E) = \{\tilde{y}_n, n = 1, 2, \dots\}$,

where $\tilde{y}_n = \sum_{k=1}^{\infty} \tilde{a}_{nk} \tilde{x}_k$ and $(\tilde{x}_k) = (\delta^k)$

Hence $\tilde{y}_n = \tilde{a}_{nk}, n = 1, 2, \dots$

By Theorem, $\tilde{A} \in (G_{\chi^2}(gI) : l(gI))$ if and only if $\tilde{A}(gI)$ is bounded in $G_{\chi^2}(gI)$ and the columns of \tilde{A} are in \tilde{G} .

$\tilde{A}(gI)$ is bounded implies there exists $M > 0$ such that $\|\{\tilde{a}_{1k}, \tilde{a}_{2k}, \dots\}\| \leq M$ for all k

That is $\left(\sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{a}_{nk}, 0)]^2\right) \leq M$ for all n

Thus $\sup\left(\sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{a}_{nk}, 0)]^2\right) < \infty, n = 1, 2, \dots$

Also the columns of \tilde{A} are in $G_{\chi^2}(gI)$

That is, $(\tilde{a}_{nk})_{k=1}^{\infty}$ belongs to $G_{\chi^2}(gI)$ and hence

$\sup\left(\sum_{k=1}^{\infty} \lambda_k^2 [d(\tilde{a}_{nk}, 0)]^2\right) < \infty, n = 1, 2, \dots$

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