International Journal of Electronics Communication and Computer Engineering
Volume 4, Issue 3, ISSN (Online): 2249–071X, ISSN (Print): 2278–4209

The General Analytical and Numerical Solution for the Nonlinear Klein-Gordon Equation

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Abstract **– We investigate the analytical solutions of the nonlinear Klein-Gordon equation with cubic nonlinearity, by applying the idea of commutative hypercomplexmathematics. And also, we apply Bernstein Spectral method to approximate the solution of the nonlinear Klein-Gordon equation. Illustrative example is included to demonstrate the validity and applicability of the technique.**

Keywords **– Nonlinear Klein-Gordon Equation, Commutative Hypercomplexatematics, Bernstein Polynomials, Collocation.**

I. Introduction

In nuclear and high energy physics the study of exact solution of the Klein-Gordon equation is of high importance for mixed scalar and vector potentials. However, the problem of exact solution of the Klein-Gordon for a number of special potential has been of great interests in the recent years. The Klein-Gordon equation plays an important role in mathematical physics [1, 2]. The equation has attracted much attention in studying solutions and condensed matter physics [3], in investigating the interaction of solutions in a collision less plasma, the recurrence of initial states, and in examining the nonlinear wave equations [4]. Different techniques have been assumed by different authors to obtain the exact solution of the Klein-Gordon with some typical potential. [5, 6, 7, 8, 9, 10]. The study of numerical solutions of the Klein Gordon and sine-Gordon equations has been investigated considerably in the last few years. [11, 12, 13, 14, 15]. In this paper we are dealing with the numerical approximation of the following nonlinear Klein-Gordon equation with cubic nonlinearity

$$u_{tt} + \alpha u_{xx} + \beta u + \gamma u^3 = f(x,t), \qquad (1)$$

Where α, β andγ are known constant. In the present paper Bernstein polynomialsand Bernstein operational matrix is presented. In this approach, a truncated Bernstein series together with the Bernstein operational matrix are used to convert problem to a system of linear algebraic equations. We demonstrate the relation between the Bernstein and Legendre polynomials. By using this relation we derive the operational matrices of derivative of Bernstein polynomials. Then we employ them for solving our problem. The remainder of this paper is organized as follows. In Section 2, we give an overview of Bernstein polynomials, and relevant properties of commutative hypercomplex mathematics that needed in the sequel. In Section 3, by using the commutative hypercomplex mathematics we obtain Analytic solutions of the nonlinear Klein-Gordon equation. In Section 4, we making Bernstein operational matrix and also we apply Bernstein Spectral method in section 5. Finally, Section 6 gives numerical result exhibiting the accuracy and efficiency of our proposed numerical algorithms.

II. Preliminaries and Notation

In the section, we give some properties of the Bernstein polynomials and also, we summarize the properties of the commutative hypercomplexmatrmatics.

A. Bernstein polynomials

Bernstein polynomials of degree m, on the interval [0, 1] as basis functions for the linear space of polynomials are defined as

$$B_{i,m} = \sum_{k=0}^{m-i} (-1)^k \binom{m}{i}\binom{m-i}{k} x^{i+k}, \qquad (2)$$

i=0,1,…,m

A polynomial h(x) of degree m canbe expressed as

$$h(x) = \sum_{i=0}^{m} b_i B_{i,m} = b^T \phi(x), \qquad (3)$$

where the Bernstein coefficient vector b and the Bernstein vector Ø(x) are given by

$$b^T = [b_0, b_1, \dots, b_m], \qquad (4)$$

and

$$\phi(x) = [B_{0,m}(x), B_{1,m}(x), \dots, B_{m,m}(x) \qquad (5)$$

The product of Bernstein polynomials is

$$B_{i,m}(x) B_{i,n}(x) = \frac{\binom{m}{i}\binom{m}{j}}{\binom{m+n}{i+j}} B_{i+j,m+n}(x), \qquad (6)$$

and

$$\int_0^1 B_{i,m}(x)dx = \frac{1}{m+1} \qquad (7)$$

For details, see [16].

The Legendre polynomials constitute an orthonormal basis on the interval [0, 1],we define as follows

$$L_0(x) = 1, \qquad L_1(x) = 2_x - 1,$$

$$L_{i+1}(x) = \frac{(2i+1)(2x-1)}{i+1} L_i(x) - \frac{i}{i+1} L_{i-1}(x),$$

i=1, 2,…

The analytic form of the shifted Legendre polynomial Li(x) of degree i given by

$$L_i(x) = \sum_{k=0}^{i} (-1)^{(i+k)} \frac{(i+k)!x^k}{(i-k)!k!^2}.$$

Note that Li(0) = (−1)i andLi(1)= 1. The orthogonality condition is

$$\int_0^1 L_i(x) L_j(x) dx = \begin{cases} 0, & \text{for } i \neq j, \\ \dfrac{1}{2i+1}, & \text{for } i = j. \end{cases}$$

A polynomial Pm(x) of degree m can be expressed as

$$P_m(x) = \sum_{j=0}^{m} l_j L_j(x) = l^T \varphi(x), \qquad (8)$$

Where the shifted Legendre coefficient vector l and the shifted Legendre vector $\varphi(x)$are given by

$$l^T = [l_0, \dots l_m],$$
$$\varphi(x) = [L_0(x), L_1(x), \dots L_m(x)]^T \qquad (9)$$

The derivative of the vector $\varphi(x)$ can be expressed by

$$\frac{d\varphi(x)}{dx} = D_l \varphi(x) \qquad (10)$$

where matrix Dl is the (m + 1) × (m + 1) operational matrix of derivative of theshifted Legendre polynomials on the interval [0,1] given by

$$D_l = d_{i+1,j+1} = \begin{cases} 2(2j+1), & \text{for } j = i-k, \\ 0, & \text{for otherwise.} \end{cases}$$

$$\begin{cases} k = 1,3, \dots m, & \text{if } m \text{ odd}, \\ k = 1,3, \dots, m-1, & \text{if } m \text{ even}, \end{cases}$$

B. Commutative hypercomplex mathematics

Systems of hypercomplex numbers, which had been studied and developed at the end of the 19th century, are nowadays quite unknown to the scientific community. Itis believed that study of their applications ended just before one of the fundamental discoveries of the 20th century, Einsteins equivalence between space and time. Owing to this equivalence not-defined quadratic forms have got concrete physical meaning and have been recently recognized to be in strong relationship with a system of bidimensional hypercomplex numbers. The commutative hypercomplex mathematics is an extension of complex numbers that obeys the axioms of the classical complex variables. It is 4-D independent variable, so we will use the notation Z = 1x + iy+jz + kct, where x, y, z, ct are real and Z belong to an element of the commutative hypercomplex algebra. In the fourth component, t represents time, and c is a scale factor [17].

Analytic function is defined as following:

$$F(Z) = F(\xi)e_1 + F(\eta)e_2,$$
$$\xi = (x - ct) + i(y + z),$$
$$\eta = (x + ct) + i(y - z),$$
$$e_1 = \left(\frac{1-k}{2}\right), \qquad e_2 = \left(\frac{1+k}{2}\right).$$

The 4-D function F(Z) is analytic if both F(ξ) and F(η) are analytic in the classical complex variable sense. Now, we introduce operators such as derivative and integral for functions of a 4-D variable. They obey the function definition that we already have. Therefore, they are as following:

Open (Z) = oper (ξ)e_1 + oper(η)e_2.

The result is that we can apply all of the powerful tools of complex analysis tofour-space problems. The 4-D Cauchy-Riemann (C-R) conditions which have a number of interesting is:

$$\frac{dF}{dZ} = 1\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y} = -j\frac{\partial F}{\partial z} = k\frac{\partial F}{\partial ct} \qquad (11)$$

C-R conditions say that the derivative ofa 4-D analytic function is the same withina sign in all four coordinate directions. The first two equalities are the same asfor complex variables. These equations can be used to reduce a partial differential equation in several real, independent variables to an ordinary differential equation in one 4-D variable. By doing so, we would be imposing continuity conditions on the PDE, because the C-R conditions are a statement of continuity. PDEs aretypically derived with the assumption of continuity, but without its explicit inclusion because convenient means have not been available. Note carefully that we are not constraining any potential solution, because the C-R conditions hold for any and all analytic functions.

III. Analytic Solution of the Klein-Gordon Equation

In this section we impose a proposition that says how to get the analytic solutions of the KleinGordon equation.

*Proposition 3.1.*The general analytical solution of Eq.(1) in 4-D space is as following:

$$u(x, ct) = \pm\frac{\sqrt{2}i}{\sqrt{b}}\left[D + \frac{(\beta\eta^2 - A\alpha)}{(\eta^2 - A)}\right] \qquad (12)$$

Where,

$$\eta = Sn\left\{\frac{\sqrt{(\beta-\delta)(\alpha-\gamma)}}{4\sqrt{3}}(x + kct - z_0), \sqrt{\frac{(\beta-\gamma)(\alpha-\delta)}{(\alpha-\gamma)(\beta-\delta)}}\right\}$$
$$A = \frac{\beta - \delta}{\alpha - \delta}.$$

In above, we have α, β, γ, δ,z0are the arbitrary constants, and k is a 4-D algebraicbasis element and c is a scale factor.

Proof. As mentioned previous section, our basic approach to solution is to first convert the (nonlinear) Klein-Gordon equation to an ODE, then solve it by means of classical methods. To convert partial differentials to ordinary derivatives, we shall use the 4-D Cauchy-Riemann equations (11), where Z = 1x + iy + jz + kct ([18]).

Making the partial derivative conversions, Eq.(1) converts to

$$k^2c^2\frac{d^2u}{dZ^2} + \mu\frac{d^2u}{dZ^2} + eu + au^3 = 0 \qquad (13)$$

By considering k2 = 1 we have:

$$(c^2 + \mu)\frac{d^2u}{dZ^2} + eu + au^3 = 0 \qquad (14)$$

This equation is still nonlinear, but is solvable by direct methods. We rearrange,multiplyboth sides of the equation by the first derivative and also let c2 + α= b, then integrate to get:

$$\frac{-b}{2}\left(\frac{du}{dz}\right)^2 = \frac{a}{4}u^4 + \frac{e}{2}u^2 + B, \qquad (15)$$

The element B is another 4-D constant of integration. We consider above equation to polynomial in u. Ames 's given the solution in this form in [19].

$$\frac{a}{4}u^4 + \frac{e}{2}u^2 + B = \frac{a}{4}[u^4 - (\alpha+\beta+\gamma+\delta)u^3 + [\alpha\beta + \gamma\delta + (\alpha+\beta)(\gamma+\delta)]u^2 - [(\alpha+\beta)\gamma\delta + (\gamma+\delta)a\beta]u + \alpha\beta\gamma\delta]$$

Then, we see

$$\alpha + \beta + \gamma + \delta = 0,$$
$$\frac{a}{4}[\alpha\beta + \gamma\delta + (a+\beta)(\gamma+\delta)] = \frac{c}{2},$$
$$\vdots$$

Now, we can use Ames 's solution for the third and final integration and we yield:

$$\pm\frac{\sqrt{2b}i}{2}u(Z) = D + \frac{(\beta\eta^2 - A\alpha)}{(\eta^2 - A)},$$

$$A = \frac{\beta-\delta}{\alpha-\delta},$$

$$\eta = \mathrm{Sn}\left\{\frac{\sqrt{(\beta-\gamma)(\alpha-\gamma)}}{4\sqrt{3}}(Z - z_0), \sqrt{\frac{(\beta-\gamma)(\alpha-\delta)}{(\alpha-\gamma)(\beta-\delta)}}\right\}.$$

The element D is the third 4-D constant of integration, z0 is arbitrary constant and the function Sn[...] is the Jacobi elliptic function. Above proposition gives us the general analytical solution for the KleinGordon equation in terms of the 4-D commutative hypercomplex variable Z. It is the complete solution for the ODE form in as much as we have integrated twice and have a solution including two arbitrary constants of integration.

We have obtained a solution u(Z) in terms of one variable having three space dimensions and time. One question that must be answered is, "Does it reduce to a solution of the original, one-dimensional KleinGordon equation when the y, z components are set to zero?", this is enough to check. Setting y = z = 0 and assuming that z0 is a constant in this space, we get:

$$u(x, ct) = \pm\left[\frac{\sqrt{2}i}{\sqrt{b}}[D + \frac{(\beta\eta^2 - A\alpha)}{(\eta^2 - A)}\right] \qquad (16)$$

Where, we have

$$\eta = \mathrm{Sn}\left\{\frac{\sqrt{(\beta-\delta)(\alpha-\gamma)}}{4\sqrt{3}}(x - Kct - z_0), \sqrt{\frac{(\beta-\gamma)(\alpha-\delta)}{(\alpha-\gamma)(\beta-\delta)}}\right\}$$

IV. Operational Matrix of Differentiation

In this section, we will try to derive an explicit formula for the (m+1)(m+1) matrix Db that is called operational matrix of derivative Bernstein polynomials. Firstly, we demonstrate the relation between the Bernstein and Legendre polynomials by introducing transformation matrix W and G. [20]. The Legendre polynomial Lk(x) can be expressed in the k − th degree Bernstein basis B0,k(x),B1,k(x), ...,Bk, k(x) as

$$L_k(x) = \sum_{i=0}^{k}(-1)^{k+i}\binom{k}{i}B_{i,k}(x) \qquad (17)$$

Now consider a polynomial Pm(x) of degree m, expressed in the m − th degree Bernstein and Legendre bases on x ∈ [0, 1] :

$$P_m(x) = \sum_{j=0}^{m} c_j B_{j,m}(x) = \sum_{k=0}^{m} l_k L_k(x) \qquad (18)$$

We write the transformation of the Legendre polynomials on [0, 1] into the m –thdegree Bernstein basis functions as

$$B_{k,m}(x) = \sum_{i=0}^{m} w_{k,i}L_i(x), \quad k = 0, \dots, m \qquad (19)$$

The elements wk,i,k,i = 0, 1, ...,m, form an (m+1)×(m+1) basis conversion matrix W. To compute them, we multiply Eq. (19) by Lj(x), integrate over x ∈ [0, 1], we have

$$w_{k,j} = (2j+1)\int_0^1 B_{k,m}(x)L_j(x)dx. \qquad (20)$$

We now replace Eq. (17) into (20) and obtain

$$w_{k,j} = (2j+1)\sum_{i=0}^{j}(-1)^{j+i}\binom{k}{i}\int_0^1 B_{k,m}(x)B_{i,j}(x)dx.$$

The integrals of the products of Bernstein basis functions can be found using

$$\int_0^1 (1-x)^r x^i dx = \frac{1}{(r+i+1)\binom{r+i}{i}}, \qquad r, i \in N \cup \{0\},$$

As follows:

$$\int_0^1 B_{k,m}(x)B_{i,j}(x)dx =$$

$$\binom{m}{k}\binom{j}{i}\int_0^1 x^{k+i}(1-x)^{m+j-k-i}dx = \frac{\binom{m}{k}\binom{j}{i}}{(m+j+1)\binom{m+j}{k+i}}$$

Therefore, we have the elements of W as

$$w_{k,j} = \frac{2_j+1}{m+j+1}\binom{m}{k}\sum_{i=0}^{j}(-1)^{j+i}\frac{\binom{j}{i}\binom{j}{i}}{\binom{m+j}{k+i}},$$

$$k, j = 0, \dots m.$$

Now, we write the transformation of the B-polynomials on [0, 1] into m-th degree
Legendre basis functions as

$$L_k(x) = \sum_{j=0}^{M} G_{k,j}B_{j,m}(x) \qquad (21)$$

$k = 0, \dots, m,$

The elements Gk,j form an(m+1)×(m+1) basis conversion matrix G. ReplacingEq.(21) into Eq. (18) and rearranging the order of summation, we obtain

$$c_j = \sum_{k=0}^{m} l_k G_{k,j}, \quad j = 0, \dots, m \qquad (22)$$

Since we can express each k-th degree Bernstein basis function in the m-th degree Bernstein basis as

$$B_{i,k} = \sum_{j=i}^{m-k+i}\frac{\binom{k}{i}\binom{m-k}{j-i}}{\binom{m}{j}}B_{j,m}(x), \qquad (23)$$

$i = 0, \dots, m, \quad i = 0, \dots, k,$

replacing Eq. (23) into Eq. (17) and re-arranging the order of summation, we find that the basis transformation (21) is defined by the elements

$$G_{k,j} = \frac{1}{\binom{m}{j}}\sum_{i=r}^{\min\{j,k\}}(-1)^{k+j}\binom{k}{i}\binom{k}{i}\binom{m-k}{j-i},$$

$$r = \max\{0, j+k-m\}, \qquad (24)$$

of the matrix G for k, j = 0, ...,m. If we denote the Legendre basis vector as using Eqs. (5), (9), (19) and (21) we have

$$\phi(x) = W\varphi(x) \qquad (25)$$

$$\varphi(x) = G\phi(X) \qquad (26)$$

$$D_b = WD_lG, \qquad (27)$$

V. Solution of the Problem

Suppose $\phi(x)$ and $\phi(t)$ are vectors of Bernstein polynomials on [0,1] defined in Eq.(5). Now the unknown function u(x, t) in Eq. (1) can be approximated as

$$u(x,t) \simeq \phi^T(x)U\phi(t), \quad (28)$$

where the unknown matrix U is (m + 1) × (m + 1) and can be shown as

$$\begin{pmatrix} U_{1,1} & \cdots & U_{1,m+1} \\ \vdots & \ddots & \vdots \\ U_{m+1,1} & \cdots & U_{m+1,m+1} \end{pmatrix}.$$

We can write

$$u_{tt}(x,t) = \left(\phi(x)\right)^T U D_b^2 \phi(t) \quad (29)$$

Also we have

$$u_{xx}(x,t) = (D_b^2\phi(x))^T U\phi(t) = \phi^T(x)(D_b^2)^T U\phi(t) \quad (30)$$

Using Eqs. (28-30) in Eq. (1), we obtain

$$(\phi(x))^T U D_b^2 \phi(t) + \alpha\phi^T(x)(D_b^2)^T U\phi(x)$$
$$+\beta\phi^T(x)U\phi(t) + \gamma(\phi^T(x)U\phi(t))^3 = f(x,t) \quad (31)$$

We now collocate Eq. (31) in (m−1)×(m−1) points (xi, tj), i = 2, ...,m, j = 2, ...,m, as

$$R(x_i,t_j) = (\phi(x_i))^T U D_b^2 \phi(t_j)\ \alpha\phi^T(x_i)(D_b^2)^T U\phi(t_j)$$
$$+\beta\phi^T(x_i)U\phi(t_j) + \gamma(\phi^T(x_i)U\phi(t_j))^3 = f(x_i,t_j) \quad (32)$$

where xi, i = 1, ...,m+1, are shifted points of Lj on the [0, 1] and tj ,j = 1, ...,m+1, are the shifted points of Lj on [0, T] totally we can choose Chebyshev collocation points instead of Legendre collocation points. [21]. Collocating initial and boundary condition in m+ 1 points xi, i = 1, ..., m+ 1, and m points tj , j = 1, ...,m, we gain

$$u(0,t_j) = f_1(t_j), \quad u(1,t_j) = f_2(t_j) \quad (33)$$
$$j = 2, \ldots, m,$$
$$u(x_i,0) = g(x_i), \quad (34)$$
$$i = 1, \ldots, m+1.$$
$$u_t(x_i,0) = h(x_i), \quad 35)$$
$$i = 1, \ldots, m+1.$$

Hence we observe that Eq. (32) together with Eqs. (33, 34) and (35) produce asystem of equations, which can be solved for the unknown Ui,j . In next section, we give some results of numerical experiments with methods based on the preceding sections.

VI. Numerical Result

In this section, for testing the accuracy and efficiency of described method we solvefollowing nonlinear example. For this aim, we consider the nonlinear Klein-Gordonequation (1) with cubic nonlinearity with constants, $\alpha = -2.5, \beta = 1$ and $\gamma = 1.5$ in the interval (x, t)∈ [0, 1] × [0, 1]. The initial conditions are given by

$$u(x,0) = B\tan(Kx), x \in [0,1] u_t(x,0)$$
$$= BcKsec^2(Kx), \ x \in [0,1],$$

and the exact solution by [22] is

$$u(x,t) = B\tan\left(K(x+ct)\right),$$

Where $B=\sqrt{\frac{\beta}{\gamma}}$ and $K = \sqrt{\frac{-\beta}{2(\alpha+c^2)}}$.

In this example f(x, t) = 0. We extract the boundary function from the exact solution. The obtained numerical results by means of the proposed method are shown in Table 1. In Table 1 and In Table 2 we compare the exact solution and approximate solution by our method for m = 9, c = 0.05 and c = 0.5. The presented results,show the efficiency and accuracy of the method.

Table 1: Absolute error for m = 9 and c = 0.05

(x_i, t_j)	m = 9
(0.1, 0.1)	−1.971 × 10−9
(0.2, 0.2)	−1.325 × 10−9
(0.3, 0.3)	3.313 × 10−10
(0.4, 0.4)	7.290 × 10−9
(0.5, 0.5)	2.140 × 10−8
(0.6, 0.6)	2.693 × 10−8
(0.7, 0.7)	3.717 × 10−8
(0.8, 0.8)	5.274 × 10−8
(0.9, 0.9)	8.767 × 10−8
(1, 1)	

Table 2: Absolute error for m = 9 and c = 0.5.

(x_i, t_j)	m = 9
(0.1, 0.1)	−1.039 × 10−9
(0.2, 0.2)	−7.048 × 10−10
(0.3, 0.3)	2.712 × 10−10
(0.4, 0.4)	3.457 × 10−9
(0.5, 0.5)	8.620 × 10−9
(0.6, 0.6)	7.594 × 10−9
(0.7, 0.7)	7.636 × 10−9
(0.8, 0.8)	7.519 × 10−9
(0.9, 0.9)	1.334 × 10−8
(1, 1)	−1.256 × 10−12

VII. Conclusions

We have considered the nonlinear Klein-Gordon equation and applied hypercomplexmathematics therefore we obtained the general analytical solution. Also, we derived operational matrix of derivative by aid of legendre polynomials as orthonormal basis.

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